Lattice Hamiltonian Truncation

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Abstract

A Hamiltonian formulation of lattice field theories can be approximated via Hamiltonian truncation. We investigate lattice Hamiltonian truncation for several low dimensional field theories and develop algorithms for defining and truncating the basis states. The lattice Hamiltonian for the Abelian Higgs model is derived, in which the gauge states are formulated as eigenstates of a rigid rotor.

1 Introduction

Nature has infinite degrees of freedom, so how can we put it on a computer? Quantum field theory (QFT) asserts that at every spacetime point there is a quantum mechanical system with a typically infinite dimensional space of states, and that this spacetime obeys the laws of special relativity. In order to compute quantities in QFT numerically, we must model the infinite degrees of freedom of the QFT with the finite degrees of freedom available to a computer. One solution, originally due to Wilson [1] for studying Quantum Chromodynamics (QCD), the theory of the strong force, is to replace the infinite, continuous spacetime with a discrete, finite one, i.e. put the theory on a lattice, typically with some finite spatial extent. In particular is the Euclidean lattice, in which time is treated like an additional real dimension, and spacetime is modeled at $\mathbb{R}^4$. The Euclidean lattice is the dominant numerical technique in high energy physics (HEP). QCD is the main application of lattice methods in HEP, since in its confined phase QCD becomes non-perturbative; the strongly coupled nature of the theory means interactions cannot be treated as a small perturbation, thus numerical methods are required.

Gauge theories are QFTs with fields that have a redundant degree of freedom. Gauge theories are ubiquitous; the Standard Model of particle physics is a gauge theory, several condensed matter systems are described by gauge theories, and physics beyond the standard model is expected to include novel kinds of gauge interactions. Despite the success of the Euclidean lattice for studying aspects of the Standard Model and in condensed matter, it only works when the Euclidean action is real and positive definite. Topological terms in gauge theories, such as the QCD theta angle, are complex in a Euclidean theory, thus using a Euclidean lattice would have a sign problem [2], thus is extremely difficult, if not impossible. Gauge theories often become non-perturbative and topological effects are of great interest in condensed matter, QCD, and beyond Standard Model physics, thus there is a need for numerical methods suited for topological terms in gauge theories.

An alternative numerical technique is Hamiltonian truncation (HT); as a generalization of the variational method in quantum mechanics, one truncates the Hilbert space at every spacetime point, for example by requiring that a given state be below a certain energy cutoff, making the Hilbert space finite dimensional. Using HT on a finite lattice gives a theory with finite degrees of freedom, which can be directly encoded on a computer. Furthermore, in the Hamiltonian approach, topological terms are real and act on the Hilbert space in a simple manner. This means HT is able to study topological terms in generic gauge theories.

To calculate a truncated Hamiltonian, we start with a local Hamiltonian operator, which is a polynomial of creation and annihilation operators. In HT one must explicitly enumerate all allowed states of the Hilbert space, and calculate the truncated Hamiltonian matrix elements in that basis. The most important states are those with the greatest weights in the Hamiltonian eigenvectors. Computational complexity scales with the number of states, so it is crucial to keep only the most important states.

In this note we review some aspect of a toy model for HT, the Abelian Higgs model, and explicitly construct its Hamiltonian in terms of creation and annihilation operators. We used this method and analogous ones to generate a finite, truncated basis of states to approximate the Hamiltonian. The results of the initial work on the Abelian Higgs model were inconclusive, thus we restrict this note to only a theoretical overview of the Abelian Higgs model and its lattice Hamiltonian.

2 Abelian Higgs Model

The Schwinger model is a 1+1 dimensional theory of Quantum Electrodynamics and is a toy model of 3+1 Quantum Chromodynamics as it exhibits confinement of fermions. We seek to calculate the Schwinger model via Hamiltonian truncation on the lattice. To understand lattice gauge theories in Hamiltonian truncation, we begin with the Abelian Higgs model, which is a theory of complex scalars with a U(1) gauge group. The Abelian Higgs model demonstrates a confinement stage, thus it resembles features of QCD, as well as exhibits many of the interesting features of a gauge theory, such as a spontaneously broken phase generated by topological effects [3].

We lay out some of the theoretical framework needed to perform Hamiltonian truncation on the Abelian Higgs model, including the gauge transformations and defining gauge fields on the lattice, then states used to construct the Hamiltonian and an algorithm for doing so, and finally the Abelian Higgs hamiltonian defined in terms of lattice operators.

2.1 Gauge Transformations

In continuous space, gauge transformations act on a single point of spacetime. On the lattice we have a complex scalar $\phi_x = \rho_x + i\chi_x$, where ρ and χ are real scalars and x denotes a position on the lattice. ϕ_x transforms as

$$\phi_x \to e^{i\theta_x}\phi_x, \tag{1}$$

where θ_x is a position dependent phase associated with the U(1) transformation. In order to construct gauge invariant operators, we associate a **link variable** $U_{x,j}$ to the connection between lattice sites at x and $x \pm j$. We will work in temporal gauge where for the gauge field A_μ, $A_0 = 0$. The gauge field will be non-dynamical, though the equations of motion will still constrain the Hilbert space. The links transform under the gauge group as

$$U_{x,j} \to e^{i\theta_x}\, U_{x,j}\, e^{-i\theta_{x+j}} = e^{i(\theta_x - \theta_{x+j})}\, U_{x,j}, \tag{2}$$

since the gauge group is Abelian.

2.2 Hamiltonian

2.2.1 Pure Gauge

In the pure gauge sector the Hamiltonian with zero potential is

$$H_0 = \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}I\left(\frac{-\partial^2}{\partial\theta^2}\right) \tag{3}$$

where there is an implicit sum over lattice sites.

The eigenstates are those of a rigid rotor with a moment of inertia I. For a state with angular momentum ℓ we write

$$\psi_\ell(\theta_x) = e^{i\ell\theta_x} \equiv |\ell_x\rangle\,. \tag{4}$$

Where ℓ_x is the link quantum number at position x. The link quantum numbers are integers that label rotor eigenstates. Eventually we will interpret $|\ell_x\rangle$ as ℓ units of flux (quanta of the gauge field) flowing from x to $x+1$, connecting the two lattice sites. Likewise, $|-\ell_x\rangle$ can be thought of as flux flowing from $x+1$ to x.

The eigenenergies are given by

$$E_\ell(\theta) = \frac{1}{2}I\ell^2. \tag{5}$$

The creation operator $\alpha^\dagger$ is defined by

$$\begin{aligned} \alpha^\dagger\,|\ell\rangle &= |\ell+1\rangle\,, \\ \alpha\,|\ell\rangle &= |\ell-1\rangle\,, \end{aligned} \tag{6}$$

from which it follows that $\alpha^{-1} = \alpha^\dagger$.

Gauge invariant states are constructed from closed loops of links, however, as then the gauge indices are fully contracted. However, in 1+1 closed loops are necessarily made of overlapping links going in the 'opposite direction'. Since the link variables are unitary, being in the $U(1)$ gauge group, a link going from the site x to $x + j$, composed with another link from $x + j$ to x gives the identity, namely $U_{x,j}U_{x,-j} = U_{x,j}U_{x,j}^{-1} = \mathbb{1}$. Thus there are no non-trivial gauge invariant states in the theory, thus it is non-dynamical. This is saying that in $1 + 1$ dimensions, the photon is not a propagating degree of freedom, rather, only a term in the Hamiltonian's potential.

2.2.2 Complex Scalar

We now turn to a theory with a scalar degree of freedom coupled to the gauge field. The gauge invariant Hamiltonian with a complex scalar is

$$H = \frac{1}{2}\sum_x |\dot{\phi_x}|^2 + |D\phi_x|^2 + m^2|\phi_x|^2 + \frac{1}{2}I\dot{\theta_x}^2. \tag{7}$$

The lattice covariant derivative is

$$D\phi_x = \frac{U_{x,1}\,\phi_{x+1} - \phi_x}{a}, \tag{8}$$

where a is the lattice spacing, which for now will taken to be 1. $U_{x,1}$ here keeps track of how the gauge field affects the dynamics of ϕ_x, and Eq [8] particularly is a difference operator that is covariant under gauge transformations, such that the entire Hamiltonian may be invariant. In following sections we will include ϕ_x into the full Abelian Higgs Hamiltonian.

2.3 States

The space of states is composed of excitations of scalar particles and antiparticles at lattice sites, and modes of the gauge field at the links. We define the scalar operators for ϕ as

$$\hat{\phi_x}\,|0\rangle = \frac{1}{\sqrt{2m}}\left(a_x + b_x^\dagger\right)|0\rangle \;= \frac{1}{\sqrt{2m}}\,|\phi_x\rangle\,, \tag{9}$$

where a_x and $b_x^\dagger$ respectively annihilate an antiparticle and create a particle, and m is the mass term in the potential of ϕ field, correspondingly it is the mass of a ϕ particle, being a local excitation of its field.

Gauge states have the form

$$|\ell\rangle = \begin{cases} \left(\alpha^\dagger\right)^\ell |0\rangle & \ell > 0 \\ (\alpha)^\ell\,|0\rangle & \ell < 0. \end{cases} \tag{10}$$

An eigenstate of the Hamiltonian defined in (7) will be a gauge invariant combination of $a^\dagger$, $b^\dagger$, $\alpha^\dagger$, and α acting on the vacuum. Note that since the link variables are eigenstates of a rotor, their quantum numbers can be negative, namely $\alpha\,|0\rangle$ is nonzero. For a state with an equal number of particles and antiparticles, whether the modes for the link variables form a gauge invariant state will be determined by the distribution of particles and antiparticles. For example, the state

$$\begin{aligned} \phi_x^\dagger\,\phi_{x+1}\,\alpha_x\,|0\rangle &= |\phi_x; -\ell_x; \bar{\phi}_{x+1}\rangle \\ &= |\phi_x; \bar{\phi}_{x+1}\rangle \end{aligned} \tag{11}$$

is gauge invariant. This can be understood as there being flux flowing from the ϕ at $x+1$ to the ϕ at x; the link quantum number at x is ℓ_x, it goes from x to $x+1$ and must be negative to denote flux leaving $x+1$ and entering at x.

We expect the energy of this state to be the sum of the frequencies of the scalar excitations, which form independent harmonic oscillators barring the gauge interactions, plus the energy of the links.

The basis for the Hamiltonian will still be specified just by the number of ϕ_x at a given lattice site, although the energy will be different. The energy of a state nonetheless is the eigenvalue of the Hamiltonian, so the question is how to write the Hamiltonian in a basis without gauge states, which still captures the gauge contributions to the potential. The algorithm to calculate the energy of the lines of flux between lattice sites would go as follows:

1. Define a state from any charge neutral combination of scalars (equal number of a and b). This will be expressed as creation and annihilation operators.

2. Associate each site with a total charge, which will be calculated as $Q_x = n_{\text{particles}} - n_{\text{anti-particles}}$.

3. Determine the link quantum number at a given site (the link at x is the one between x and $x+1$) by $\ell_x = Q_{x+1} - Q_x$.

4. The energy of the state is the energy as calculated by acting with the gauge free Hamiltonian, plus $\sum_x \frac{1}{2} I \ell_x^2$.

This will work for calculating the energy of the state, but it only works for eigenstates of the Hamiltonian. The goal is to perform Hamiltonian truncation, so we must be able to calculate matrix elements of H generally. One idea is since the link variables are determined by the distributions of scalars, we could include an interaction term

$$\begin{aligned} H_i &= \sum_x \frac{1}{2} I \hat{\ell}_x^2, \\ \hat{\ell} &= \hat{Q}_{x+1} - \hat{Q}_x, \\ \hat{Q}_x &= \hat{n}_b - \hat{n}_a, \\ \hat{n}_{b,x} &= b_x^\dagger b_x, \\ \hat{n}_{a,x} &= a_x^\dagger a_x. \end{aligned} \tag{12}$$

This is the same procedure as above, except instead of explicitly counting the charge of a state, we insert number operators into the Hamiltonian.

2.4 Abelian Higgs Hamiltonian

The Hamiltonian in the 1D abelian Higgs model is

$$\begin{aligned} H &= H_{\text{gauge}} + H_{\text{scalar}}, \\ &= H_{\text{gauge}} + H_{0,\text{scalar}} + V_{\text{scalar}}, \end{aligned} \tag{13}$$

where

$$\begin{aligned} H_{\text{gauge}} &= E_{\text{vacuum}} + \sum_x \frac{1}{2} g^2 E_x^2 - \frac{g^2\theta}{2\pi} E_x, \\ H_{0,\text{scalar}} &= \sum_x |\dot{\phi_x}|^2 + \overline{m}^2 |\phi_x|^2, \\ V_{\text{scalar}} &= \sum_x |U_x \phi_{x+1} - \phi_x|^2 + (m^2 - \overline{m}^2)|\phi_x|^2 + \frac{\lambda}{8}|\phi_x|^4. \end{aligned}$$

The vacuum energy $E_{\text{vacuum}} = -L\frac{g^2\theta^2}{8\pi^2}$, L is the size of the lattice, and g is the gauge coupling.

ϕ_x is a complex scalar at the lattice site x given by

$$\begin{aligned} \phi_x &= \frac{1}{\sqrt{2m}} \left(b_x + a_x^\dagger\right), \\ \dot{\phi}_x &= \sqrt{\frac{m}{2}} \left(b_x - a_x^\dagger\right), \end{aligned} \tag{14}$$

where $a_x^\dagger, b_x^\dagger$ are creation operators for ϕ particles and antiparticles, respectively. The link variable U_x is the gauge connection between lattice sites x and $x+1$, and can be thought of as a rigid rotor where

$$U_x = e^{iA_x}, \tag{15}$$

where A_x is the photon field.

E_x is the electric field and is given by

$$E_x = \frac{\partial \mathcal{L}}{\partial \dot{A}_x} = \frac{1}{2} g^2 \dot{A}_x + \frac{\theta}{2\pi}. \tag{16}$$

2.4.1 Scalar Hamiltonian

We can write the Hamiltonian of the scalar sector as a sum of monomials:

$$
\begin{aligned}
H_{0,x} &= \overline{m}\left(X_1 + Y_1 + \frac{1}{2}\right) \\
\sum_x |U_x\phi_{x+1} - \phi_x|^2 &= \frac{1}{\overline{m}}\sum_x X_1 + Y_1 + Z_1 + Z_1^\dagger + 1 \\
&\quad - \frac{1}{2}\left(G_1 + G_2 + G_3 + G_4 + \text{h.c.}\right), \\
(m^2 - \overline{m}^2)|\phi_x|^2 &= \frac{m^2 - \overline{m}^2}{2\overline{m}}(X_1 + Y_1 + Z_1 + Z_1^\dagger + 1), \\
\frac{\lambda}{8}|\phi_x|^4 &= \frac{\lambda}{32}(3X_1 + 3Y_1 + X_2 + Y_2 + 3Z_3 \\
&\quad + [3Z_1 + Z_2 + 2Z_4 + 2Z_5 + \text{h.c.}]),
\end{aligned}
\tag{17}
$$

where

$$
\begin{array}{lll}
X_1 = a_x^\dagger a_x & Z_1 = a_x b_x & G_1 = U_x a_x^\dagger a_{x+1} \\
X_2 = a_x^\dagger a_x^\dagger a_x a_x & Z_2 = a_x a_x b_x b_x & G_2 = U_x b_{x+1}^\dagger b_x \\
Y_1 = b_x^\dagger b_x & Z_3 = a_x^\dagger b_x^\dagger a_x b_x & G_3 = U_x a_{x+1} b_{x+1} \\
Y_2 = b_x^\dagger b_x^\dagger b_x b_x & Z_4 = a_x^\dagger a_x a_x b_x & G_4 = U_x^{-1} a_x b_{x+1} \\
 & Z_5 = b_x^\dagger a_x b_x b_x &
\end{array}
$$

The full Hamiltonian is then

$$
\begin{aligned}
H = \sum_x \, & \left(\frac{m^2 + \overline{m}^2 + 2}{2\overline{m}} + \frac{3\lambda}{32}\right)(X_1 + Y_1) + \frac{\lambda}{32}(X_2 + Y_2) \\
& + \left(\frac{m^2 - \overline{m}^2 + 2}{2\overline{m}} + \frac{3\lambda}{32}\right) Z_1 + \frac{3\lambda}{32} Z_3 + \frac{\lambda}{32}(Z_2 + 2Z_4 + 2Z_5 + \text{h.c.}) \\
& - \frac{1}{2}(G_1 + G_2 + G_3 + G_4) \\
& + \frac{m^2 + 2}{\overline{m}}.
\end{aligned}
\tag{18}
$$

2.4.2 $g \to 0$ limit

The Lagrangian for the fields used in 13 is

$$
\mathcal{L} = \frac{1}{2g^2} F_{01}^2 + \frac{\theta}{2\pi} F_{01}, \tag{19}
$$

where $F_{01} = \dot{A}_x$ which is singular for $g \to 0$. We can avoid this by rescaling the field $\tilde{A}_x = gA_x$. The Hamiltonian in the pure gauge sector becomes

$$
H_{\text{gauge}} = E_{\text{vacuum}} + \sum_x \frac{1}{2}\tilde{E}_x^2 - \frac{g\theta}{2\pi}\tilde{E}_x, \tag{20}
$$

where $\tilde{E}_x = \dot{\tilde{A}}_x + \frac{g\theta}{2\pi}$. The electric field still acts as as a derivative with respect to the rescaled photon field, thus the eigenstates of the rescaled Hamiltonian are

$$\psi_s(\tilde{A}) = \prod_x \frac{1}{\sqrt{2\pi}} e^{igs_x\tilde{A}_x}, \tag{21}$$

where s_x is the number of photon excitations at x. The rescaled connection acts as

$$U_x \left|\ldots, s_x, \ldots\right\rangle = e^{igs_x\tilde{A}_x} \prod_x \frac{1}{\sqrt{2\pi}} e^{igs_x\tilde{A}_x} = \left|\ldots, s_x + 1, \ldots\right\rangle, \tag{22}$$

where is simply a ladder operator, as before the rescaling. We find that the eigenvalues are unaffected by the rescaling.

3 Conclusion

The lattice Hamiltonian defined the theory, then in order to perform Hamiltonian truncation we must express the Hamiltonian in terms of its lattice degrees of freedom. We give the lattice Hamiltonian for a massive complex scalar and massless gauge field in a 1+1 dimensional Abelian Higgs theory in terms of creation and annihilation operators. Current and future work implement Hamiltonian truncation to calculate ground state energies and other physical quantities, and tests for the convergence and efficiency thereof. A truncated basis can be used to construct an approximate Hamiltonian. This can be used to approximate the spectrum of the theory, and this approximation becomes more accurate as the basis grows larger. A truncated basis grows rapidly with the parameters used to truncate it, here the number of scalar and link excitations, and it is pivotal to keep only the states which contribute most substantially to the Hamiltonian.

References

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