

Single Vortex Motion of Superfluid Helium

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Abstract—The work presented was done in a summer REU program hosted by the University of California, Davis. The goal is to better understand the single vortex motion of superfluid helium when the vortex is in a container that has two cylindrical segments of different diameters, joined by a short sloped portion.

I. INTRODUCTION

There are two types of classical vortices, a free vortex, and a forced vortex. A forced vortex is when there are external forces that are acting on a fluid and making the fluid rotate. All fluid particles move at a constant angular velocity. A free vortex, which is also known as an irrotational, is formed and there is no external force or energy that is required to rotate the fluid. The overall energy flow of a free vortex remains constant and the fluid particles do not rotate about their own axis. There are many common examples of free vortices in nature such as whirlpools and tornadoes.

A superfluid is a rare state when a fluid has zero viscosity meaning that it has zero resistance to the flow of mass. It is a quantum effect that only happens at low temperatures. For instance, Helium-4 is a superfluid at low temperatures, at about 2 Kelvin.

Superfluid vortices differ from classical irrotational vortices since they are quantized, it simply means that at certain distances away from the core there are only certain speeds the fluid can travel. There is some minimum velocity for the fluid and at all points along the vortex, the fluid must be traveling an integer multiple of the minimum velocity. For classical vortices, the vortex dissipates energy, as a result, the fluid flow slows down and the circulation of the vortex decreases. This does not happen for a superfluid vortex due to the fact it is quantized. The superfluid velocity depends on how far away you are from the core. Therefore, the superfluid that is closer to the core will move at a higher velocity than the superfluid that is further from the core. It is important to study the vortex motion of superfluid helium and learn more about its properties in hopes to better understand classical vortices and other areas of physics. To study the single vortex motion of superfluid helium the velocity field produced from the superfluid helium vortex must be analyzed.

II. TECHICAL BACKGROUND

Superfluid vortices are irrotational vortices, meaning that the curl of the velocity field (\vec{V}) is equal to zero, $\nabla \times \vec{V} = 0$ (where $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$). This implies that there exists a velocity potential function φ such that $\vec{V} = \nabla\varphi$. It is also

true that $\nabla \cdot \vec{V} = 0$. By using these equations we can derive Laplace's Equation.

$$\nabla \cdot \vec{V} = \nabla \cdot (\nabla\varphi) = \nabla^2\varphi = 0$$

Due to the fact that we are specifically looking at superfluid vortices we can solve for the velocity field of the vortex by using Laplace's equation:

$$\nabla^2\varphi = 0 \quad (1)$$

where ∇^2 is known as the Laplacian Operator and is defined as $\nabla^2 = \frac{\partial^2}{\partial x^2}\hat{i} + \frac{\partial^2}{\partial y^2}\hat{j} + \frac{\partial^2}{\partial z^2}\hat{k}$. It is simple to solve Laplace's equation for a scalar field as $\nabla^2\varphi = \nabla \cdot \nabla\varphi = 0$ but becomes more complicated to solve for a vector field. With a vector field, each component of the vector field must also satisfy Laplace's equation. Hence for a vector field, Laplace's equation simplifies to $\nabla^2\vec{V} = \nabla^2V_x\hat{i} + \nabla^2V_y\hat{j} + \nabla^2V_z\hat{k} = 0$. There are a few important theorems associated with Laplace's equation, for example, the Superposition Theorem, which states that if $\varphi_1, \varphi_2, \varphi_3, \dots, \varphi_n$ are solutions to Laplace's equation then $\varphi = C_1\varphi_1 + C_2\varphi_2 + C_3\varphi_3 + \dots + C_n\varphi_n$ (where $C_1, C_2, C_3, \dots, C_n$ are defined as constants) is also a solution. As a result, we can uniquely identify a vector field (in our case the velocity field of the vortex) by solving Laplace's equation with specific boundary conditions. Since it's too complicated to solve Laplace's equation by hand for the velocity field we used a software called "depSolver" instead. depSolver is an element boundary solver for Laplace's equation in 3D. (More information about depSolver can be found at <https://depsolver.readthedocs.io/en/latest/>) To use depSolver the user must create a mesh, which is usually the surface of a 3D shape. The desired mesh can be any arbitrary shape that the user wants, it can be as simple as a cube or any more complicated shape. depSolver requires very specific input files, such as, one that creates the desired mesh, one that provides the boundary conditions that are used to solve Laplace's equation, and one that lists all points within the mesh you want to solve Laplace's equation at. depSolver produces output files that describe the value of each component of velocity at all internal points within the mesh.

III. RESULTS

A. Cylinder with no vortex

We started with a simple case, involving no vortex. We created a cylinder that is centered on the origin with radius one and height one. Figure 1 depicts the cylinder mesh used.

Each dot represents a point on the mesh. To keep the system simple we made the velocity field to be constant and equal to +5.0 only in the x-direction. Now the y-component and the z-component of the applied velocity field are both equal to zero ($V_y = 0$ and $V_z = 0$) and the applied velocity field $\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$ can be simplified to $\vec{V} = V_x\hat{i}$. Hence the velocity field would flow from the left to right through the cylinder, this system is unphysical, but we wanted to create an easy case to ensure that we understood the inputs and outputs of depSolver properly.

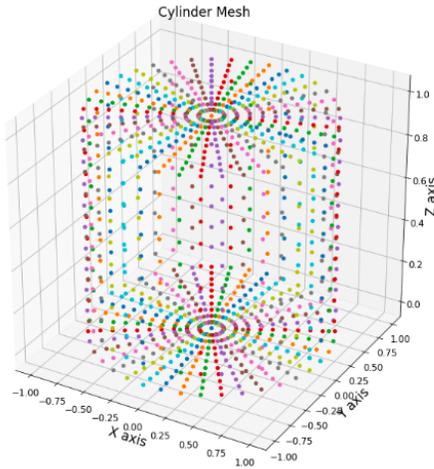


Fig. 1. Cylinder Mesh

After designing the mesh, we had to create the boundary condition file which involved solving for the perpendicular component of the velocity at all points along the surface of the mesh. We are specifically looking at the perpendicular component of velocity as it must vanish at the wall of the cylinder. If it doesn't vanish at the walls of the mesh then it implies the fluid is flowing through the wall which can't physically happen. We find that the perpendicular component of velocity for all points along the surface of the cylinder has two components, an x-component and a y-component. This is because the cylinder is a curved surface. We can simplify the perpendicular component of the velocity field by using the geometry of a cylinder: $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$, and $r = \sqrt{x^2 + y^2}$.

$$\begin{aligned} (\vec{V} \cdot \hat{n})\hat{n} &= (V_x \cos\theta)(\cos\theta\hat{i} + \sin\theta\hat{j}) \\ &= V_x \cos^2\theta\hat{i} + V_x \cos\theta\sin\theta\hat{j} \\ &= V_x \left(\frac{x^2}{x^2 + y^2}\right)\hat{i} + V_x \left(\frac{xy}{x^2 + y^2}\right)\hat{j} \end{aligned}$$

There is no velocity field at the top or bottom base of the cylinder (when $z = 0.0$ and $z = 1.0$) since the field is parallel to the bases so the z-component of the perpendicular component is zero. Having solved for the perpendicular component of the velocity field, we can then create the boundary conditions file and thus run depSolver. Figures 2 and 3 represent the total velocity field of the cylinder at specific z-values on the XY

plane. The arrows represent the magnitude and direction of velocity at each point. Figure 2 is at $z=0.5$, in the middle

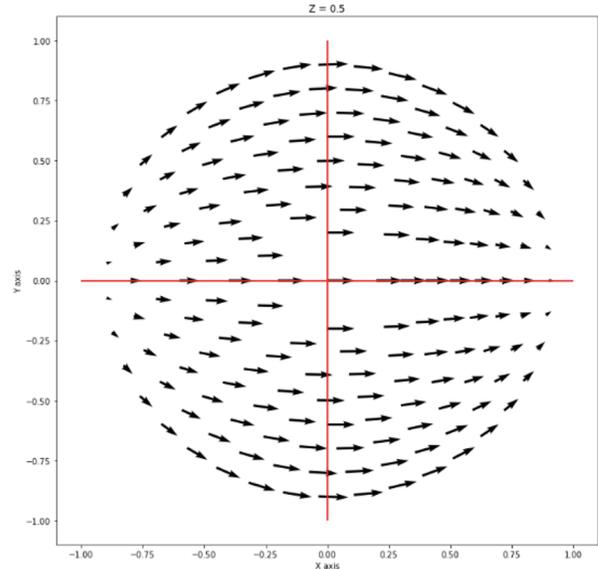


Fig. 2. Velocity field at $z = 0.5$

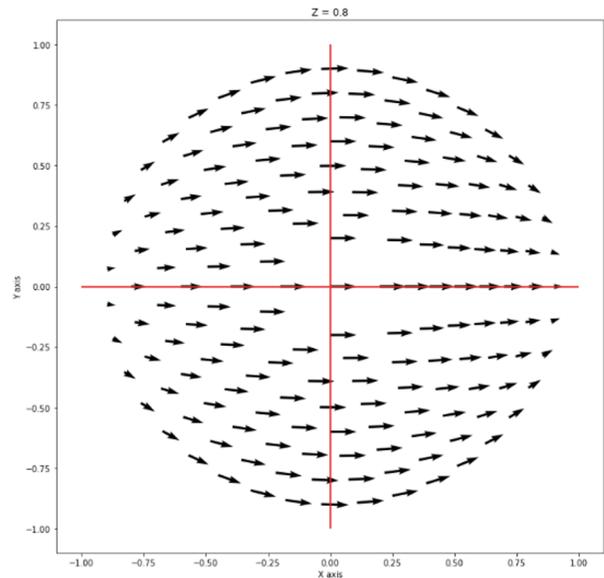


Fig. 3. Velocity field at $z = 0.8$

of the cylinder and Figure 3 is at $z=0.8$, towards the end of the cylinder. The fields look very similar to each other but not identical. We can attribute this due to the fact that $z=0.8$ is closer to the top and will have slight deviations due to the effect from the top and how the top of the cylinder is flat. We would see a similar influence if instead, we were closer to the base of the cylinder. However, the ends of the cylinder do not influence the middle of the mesh as it is far enough away from the ends and that's the area we are most interested in so the effect from the bases can be ignored. The velocity

field is tangential to the boundary almost everywhere except when $y = 0.0$, at the points $(-1.0, 0.0)$ and $(1.0, 0.0)$. This can be contributed to the fact that the system we have created is somewhat unphysical. The field flows perfectly in the x -direction with a constant magnitude everywhere but within a closed circular container, the field would be unable to flow perfectly all in the same direction with constant magnitude. After ensuring our understanding of depSolver we then moved on to the more complicated case of placing a vortex within the mesh.

B. Mesh with vortex

Figure 4 represents the vortices we are putting in the mesh. The dashed line represents the core of the vortex and then at some point, in this picture the point is represented by A, the vortex goes off at a certain angle on the XY plane and hits the wall of the container. The superfluid is rotating around the core counterclockwise and is notated by the arrows on the circles. We have also created a new mesh, the new mesh is

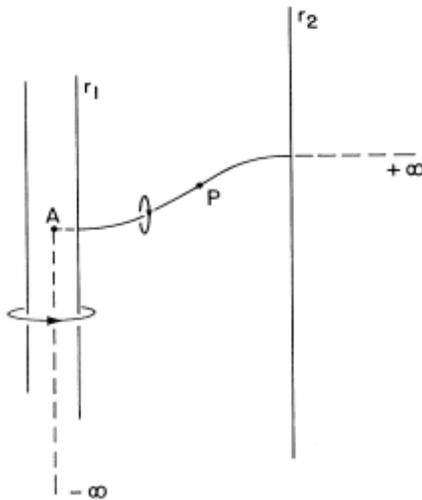


Fig. 4. Vortex Path [1]

composed of two cylinders of two different radii connected by a short sloped portion. The cylindrical mesh is depicted in Figure 6. The radius of the smaller cylinder is 0.3, the radius of the bigger cylinder is 0.6, and the height of the cell is now 5. We are specifically using this set up because it is similar to the apparatus that is used in the physical experiments involving superfluid vortices. Due to the presence of the vortex calculating the perpendicular component of the velocity at every point along the mesh becomes more difficult. I am continuing the previous work of two students and one of the previous students wrote a program that computes the perpendicular velocity at all points along the surface of the mesh and the velocity at all internal points due to the vortex. The perpendicular component velocity at all internal points is computed using the Biot-Savart Law and integrating over the vortex. The perpendicular component of velocity from the

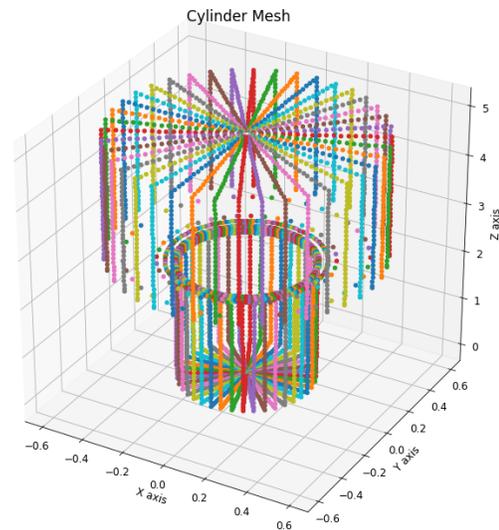


Fig. 5. Cylindrical Mesh

vortex is then used as the boundary conditions in solving for Laplace's Equation.

After designing the new mesh we then picked out vortices that are at about every 30 degrees. Figure 7 depicts the vortex

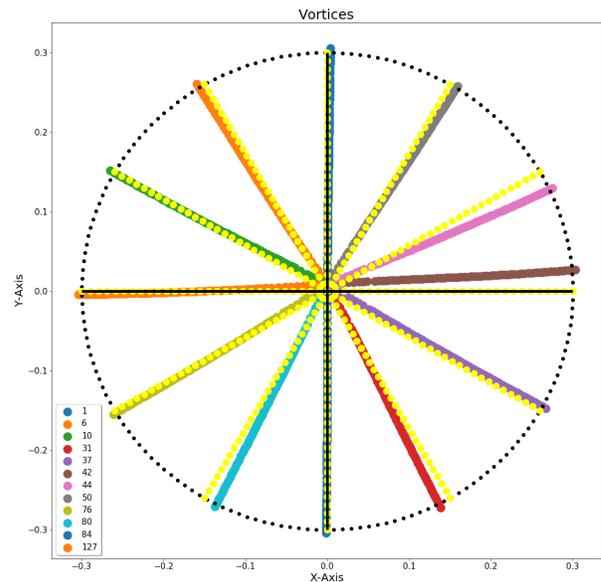


Fig. 6. Vortex Coordinates

files we worked with. The yellow dots represent exactly every 30 degrees in a circle and the colored lines represent the vortex files we had. The data points for each vortex file were taken from vortex simulations done in a perfect cylinder. It is important to note that all of the vortices are slightly off-center, the vortex core is located at $x = 0.0$ and $y = 0.009$. This can be seen in Figure 8. All of the vortices end at the same spot, at $z=2.40$. This is right before the slanted region, the slanted region begins at $z=2.41$. All of the vortices end in

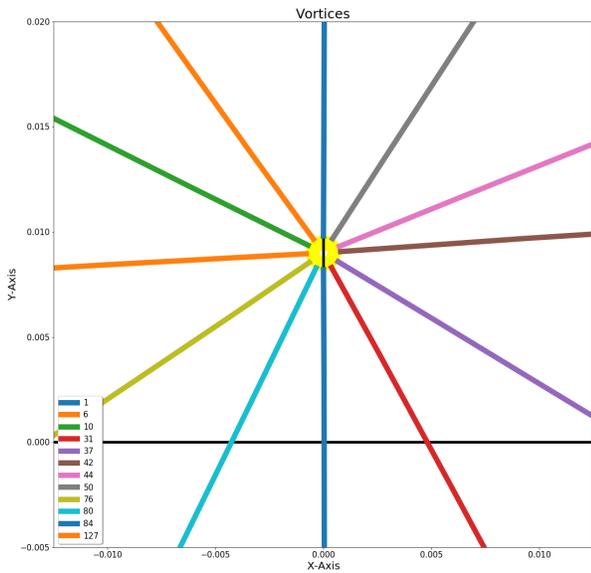


Fig. 7. Close up of Vortex Coordinates

the lower cylinder of the mesh. Figure 9 depicts the vortex

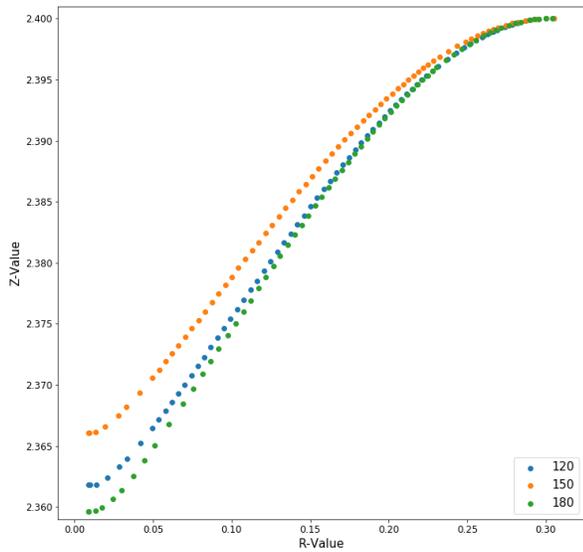


Fig. 8. Vortices going off to hit the wall of the lower cylinder

coordinates of the 120-degree, 150-degree, and 180-degree vortices from the moment the vortex goes off to hit the wall of the lower cylinder. By doing every thirty degrees for the off-center wire we can see how the velocity fields compare to one each other. The goal is to discover if the velocity fields are essentially the same just rotated or different from each other. Since the vortex core is off-center, it isn't exactly clear how the velocity fields compare. The vortices are not identical, the vortices have slightly different lengths. For example, the vortex at 270-degrees is slightly longer than the vortex located at 90-degrees. If the vortex core was on center, the vortices would be identical and the velocity fields for the vortices would be

the same but at different angles.

After selecting the vortex files we were then able to run depSolver for the different vortices. Figure 10 illustrates the XY plane of the 120-degree vortex at the height of $z = 2.37245$ within the mesh. The red vectors represent the perpendicular

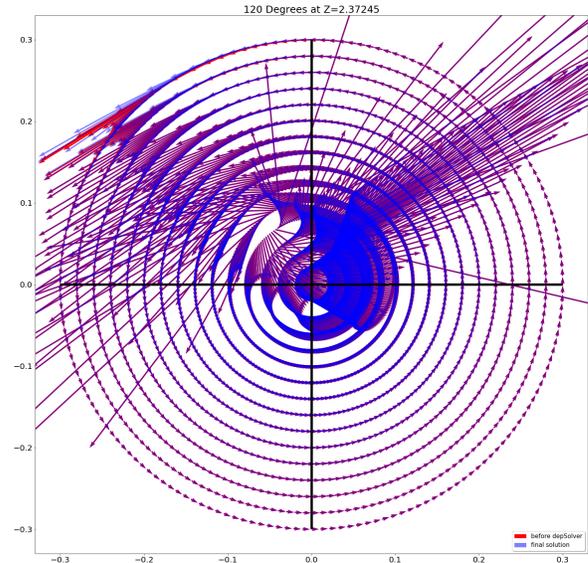


Fig. 9. Vortex 120 at $Z=2.37245$

component of velocity that we solved for at every internal point within the mesh. The blue vectors represent the total velocity field formed by the vortex. The total velocity field is calculated by taking the field we solved for and subtract the field depSolver solved for. Thus it takes into consideration the correction of depSolver, to ensure the field vanishes at the wall of the mesh. Most of the graph is purple because of the overlap between the red field and the blue field, at those points the red and blue fields are indistinguishable. Figure 11 represents the XY plane of the 120-degree vortex, 150-degree vortex, and 180-degree vortex at the height of $z = 2.37245$ within the mesh. The velocity fields at each angle are essentially the same, purely rotated from one another. Figure 11 illustrates the 120-degree, 150-degree, and 180-degree vortices at $z = 3.7179$, in the upper cylinder portion of the mesh. At each point within the mesh the velocity fields produced from the vortices are essentially the same, merely at different angles. Similar results have been reproduced in other trials with different vortices.

IV. CONCLUSION

So now that we have seen that the velocities from vortices at different angles for an off-center wire are pretty similar the next step would be to figure out how to properly rotate one of the velocity fields, let's say the velocity field at 120-degrees to 180-degrees and compare again. Another step is to run trials where instead of having the vortices end in the lower cylinder have them end in the slanted region or the upper cylinder. The goal is that hopefully, in the end, we can build a table of boundary conditions for vortices at different

REFERENCES

- [1] K.W. Schwarz. "Unwinding of a single quantized vortex from a wire," Physical Review B, vol. 47, num. 18, May 1993.

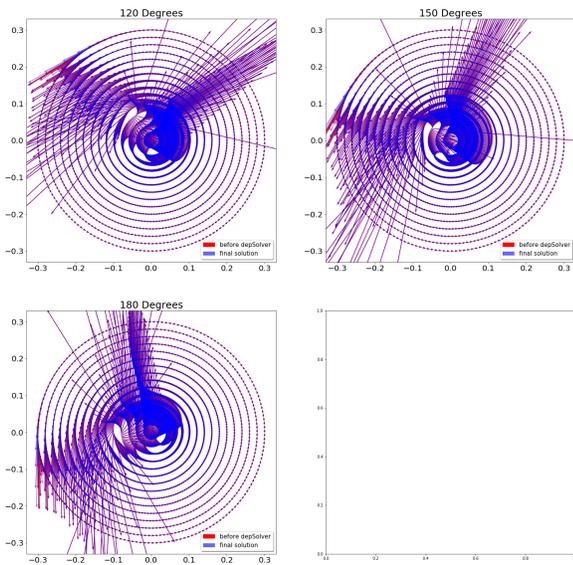


Fig. 10. Vortices 120, 150, 180 at $Z=2.37245$

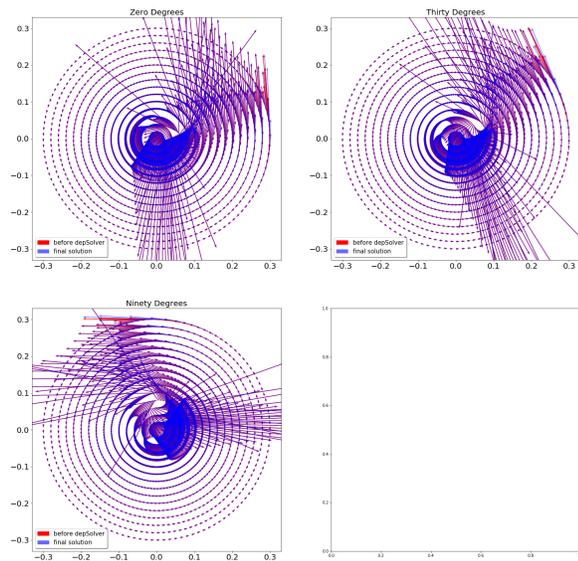


Fig. 11. Vortices 120, 150, 180 at $Z=3.7179$

angles so that we can use those conditions in the single vortex simulations. Through these simulations and mimicking what is done in physical experiments, we can use these computational results to better understand what is happening in the physical experiments. This is important so we can study how the boundary conditions change when the vortices end in different areas of the mesh, specifically in areas where the diameter changes.

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