Perturbatively Calculating Thermal Out-of-Time-Ordered Correlators for the Anharmonic Oscillator

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Abstract

Out-of-Time Order Correlators (OTOC's) are important quantities of interest in many quantum field theories, and arise in the context ranging from linear response to chaos in AdS/CFT correspondence. Until now, there has not been a systematic study of calculating OTOC's perturbatively given that most standard methods only apply to time-ordered correlation functions. Here, we look at the quantum anharmonic oscillator (i.e. scalar field theory in 0+1 dimensions), whose Hamiltonian is given by $\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\hat{x}^2 + \epsilon\hat{x}^4$. We provide a scheme to calculate any correlation function of $\hat{x}$ operators to any order in ϵ in a thermal distribution at an inverse temperature β. We explain drawbacks of the perturbation theory and possible future routes of study.

1 Introduction

1.1 What is an OTOC?

Correlation functions are quantities in quantum field theories that are used to characterize the theory and to calculate quantities of interest. For example, for a quantum mechanical system with a Hamiltonian $\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\hat{x}^2 + V(x)$, a quantity of interest could be the product of position operators at different times $\langle x(t_1)x(t_2)\rangle$ (here, we are working in the Heisenberg representation where $x(t) = e^{iHt}x(0)e^{-iHt}$ and quantum states stay fixed in time).

In quantum field theories, a formalism that facilitates the calculation of these correlators is the path integral (a good detailed exposition is given in [1]). In short, the time evolution operator e^{-iHt} can be represented by a "sum over histories", where the amplitude to arrive at point x_2 starting from point x_1 after a time t is given as.

$$\langle y|e^{-iHt}|x\rangle = \int [Dx_\mu] e^{iS[x_\mu]} \tag{1}$$

Where the integration measure $[Dx_\mu]$ is the sum over all trajectories starting at (spatial position) x and arriving at y after time t, and $S[x_\mu]$ is the classical action along the path.

In a similar way, correlation functions can be calculated in this formalism. However, a caveat is that the standard path integral formalism only allows for 'time-ordered' correlation functions $\langle Tx(t_1)...x(t_n)\rangle$ where the symbol T orders the operators by time in ascending order. For example $\langle Tx(4)x(-2)x(1)\rangle = \langle x(-2)x(1)x(4)\rangle$. The formula for correlation functions in the ground state is given as

$$\langle Tx(t_1)...x(t_n)\rangle = \frac{1}{Z}\int [Dx_\mu] x(t_1)...x(t_n) e^{iS[x_\mu]}$$

where the 'partition function' is

$$Z = \int [Dx_\mu] e^{iS[x_\mu]}$$

and the integration runs over all trajectories starting at $t = -\infty$ and ending at $t = \infty$. (This result is sometimes referred to as the Gell-Man Low Theorem).

So, time ordered correlation functions are well-captured by the path integral formalism. But, if we want to compute OTOC's, which are correlators that are not time-ordered, the situation becomes more tricky.

1.2 Why do we care about OTOC's?

OTOC's show up in various different contexts. For example, they are used to study linear response functions, which measure the change in measurable quantities under perturbations V of the Hamiltonian $H \to H + V$. To measure an operator A under the perturbations, the correlation functions $\langle [A(t), V(t')]\rangle$ and $\langle \{A(t), V(t')\}\rangle$ are often found in the formulas for the linear response.

In more recent years, OTOC's have come up as important quantities in describing the chaotic behavior of black holes and the AdS/CFT correspondence. In particular, the OTOC

$$C(t) = -\left\langle [x(t), p(0)]^2 \right\rangle$$

called the chaos correlator, is an important object of study.

Let us note some properties of the two OTOC's that we've described above. For the first one, $\langle [A(t), V(0)] \rangle$, let us note that for the term $\langle A(t)V(0) \rangle$ the order of times of the correlators looks like $(t \rightarrow 0)$ [1]. Assuming $t > 0$, this differs from the time order of $(0 \rightarrow t)$. For the chaos correlator, the term $\langle x(t)x(0)x(t)x(0) \rangle$ will go like $(t \rightarrow 0 \rightarrow t \rightarrow 0)$. This means that the chaos correlator goes back and forth in time more than once, whereas the response function correlators only do that once. In this sense, the chaos correlator has more out-of-time loops and is more out of time than previous ones. This classification of OTOC's in terms of how often they go out of time is done in more detail in [2][3].

1.3 Some (terse) Math background

Let's review a few mathematical concepts needed to talk about OTOC's. Here, we will only review the concepts from a bird's eye view, since detailed discussions and derivations would take many pages. But, we will include references to relevant textbooks and papers that have more detail.

1.3.1 Free theories and Perturbation theory

One important property of quantum field theories is that certain classes of theories known as free field theories are exactly solveable in the sense that we can derive expressions for all of their correlation functions. For example, the harmonic oscillator Hamiltonian $H = \frac{1}{2}p^2 + \frac{1}{2}x^2$ is exactly solveable, since for these theories, $x(t) = x(0)cos(t) + p(0)sin(t)$ and $p(t) = \dot{x}(t)$.

It is also possible to calculate small perturbations of these theories using perturbation theory. For example, if we want to solve for the theory $H = \frac{1}{2}p^2 + \frac{1}{2}x^2 + \epsilon x^4$ for small ϵ, we can use the path integral to get expressions for correlation functions. Some of these computations are shown below in the section "All Orders Calculation", but they are standard calculations that can be found in most QFT books, including [1].

[1] We're not taking limits. Arrow notation throughout this essay will usually refer to time orderings, except where clear from context

1.3.2 Imaginary time and Temperature

It is a well known fact that the thermal density matrix at a temperature $T = \frac{1}{\beta}$ is given by $\rho = \frac{1}{Z}e^{-\beta H}$. It is also well known that this is almost exactly the same formula for the time evolution operator e^{-iHt} but with β playing the role of imaginary time, $\beta = -it$.

So, the partition function for this matrix can be expressed in the path integral language. So,

$$Z = \sum_x \langle x|e^{-\beta H}|x\rangle$$

can be interpreted in terms of (1) as the sum of all paths starting and ending at the same point, over all trajectories initial starting points 'traversing an imaginary time β'. This allows us to calculate thermal correlation functions, averages of operators with respect to that thermal distribution. For example, the correlation function

$$\langle x(t_1)x(t_2)\rangle_\beta = \frac{1}{Z}\sum_y \langle y|x(t_1)x(t_2)e^{-\beta H}|y\rangle$$

can be interpreted as measuring the average value of $x(t_1)x(t_2)$ along all time contours going from $0 \to t_1 \to t_2 \to -i\beta$. The same path integral analogy can be extended to more general OTOC's. (More detailed discussions of the time contours can be found in [1][2][3]. Here we are not mentioning some technicalities about how to perform the path integral in imaginary time).

These OTOC's with respect to a thermal distribution are what we studied.

1.4 What we did over the summer

The work we did over the summer consisted of a couple of different things. Much of the work happened in collaboration when R. Loganayagam (a collaborator of Mukund's) coincidentally gave this same problem to his summer student Anish Kulkarni.

The first thing Mukund and I did was write programs to numerically find some OTOC's using Mathematica, with techniques we learned from [5]. From Loga and Anish, we learned that the perturbative techniques we used break down for large times. For example, they found that using standard path-integral perturbative gave $|| \langle x(t)x(0)\rangle_\beta || \to \infty$ as $t \to \infty$. This is unphysical since for an oscillator, the expectation values of x should be bounded as $t \to \infty$. They found that these are due to so-called secular terms that show up even in perturbation theory of classical oscillatory systems (e.g. similar terms show up when pertubatively solving the anharmonic oscillator's classical equations of motion). They found a paper by Bender & Bettencourt [4] that introduced perturbative techniques for the $\hat{x}$ operator that avoided these so-called secularities and remained well-behaved as $t \to \infty$. Many of our numerical comparisons matched up well with the results obtained by using the formulas in that paper.

Also, I spent a lot of time working through various text books and online resources that explained how to do the standard perturbation theory for quantum systems, including the techniques being explored by Mukund and collaborators called Schwinger-Keldysh techniques (see [2] for explanations and references). At the end I found a method to compute any correlator to any order in perturbation theory that did not rely on the Schwinger-Keldysh formalism and coded the procedure in Mathematica. The results ended up mathching exactly with Anish's results. This is discussed and derived in the section "All Orders Calculation".

1.5 Future research directions

For future work, there are many avenues to explore. For example, the calculation technique I found does not get rid of the secular terms. Work is being done by Anish and Loga to find out how to do the computations in a way that avoid these terms.

Also, it would be ideal to try to figure out how to do the computations for more complicated perturbative theories. The quantum anharmonic oscillator is the simplest example of a perturbative quantum theory. In the future, we could ask whether or not these calculations could be generalized to quantum field theories, including for fermionic theories, gauge theories, and theories with more complicated interactions. It is not clear whether the methods we found for the anharmonic oscillator suffice for these more complicated theories, or whether we need to use Schwinger-Keldysh methods for these systems too.

If the computations can be generalized, then new windows could be opened up for quantum chaos theory and the other fields that use OTOC's. New calculational techniques could reveal new phenomena and clarify concepts in a way that are now out of reach.

2 All Orders Calculation

Here, I will explain how I did the calculations for the corrections to the Wightman functions $\langle X(t_1)...X(t_n)\rangle_\beta^{perturbed}$. I will present and prove the main formula that I use. We will assume that the perturbed Hamiltonian is given by $H = \frac{1}{2}p^2 + \frac{1}{2}x^2 + \epsilon x^4$

Lemma 1. *The first order correction to the n-point Wightman function $\langle X(t_1)...X(t_n)\rangle_\beta$ is given by:*

$$
\begin{aligned}
\langle X(t_1)...X(t_n)\rangle_\beta^{O(\epsilon)} = \frac{Z_{free}}{Z^{O(\epsilon)}}\Bigg(&\langle X(t_1)...X(t_n)\rangle_\beta + \\
& i\epsilon\Bigg(\int_0^{t_1} d\tau \, \langle X^4(\tau)X(t_1)X(t_2)...X(t_n)\rangle_\beta^{free} \\
& + \int_{t_1}^{t_2} d\tau \, \langle X(t_1)X^4(\tau)X(t_2)...X(t_n)\rangle_\beta^{free} \\
& + ... \\
& + \int_{t_{n-1}}^{t_n} d\tau \, \langle X(t_1)X(t_2)...X^4(\tau)X(t_n)\rangle_\beta^{free} \\
& + \int_{t_n}^{i\beta} d\tau \, \langle X(t_1)X(t_2)...X(t_n)X^4(\tau)\rangle_\beta^{free}\Bigg)\Bigg)
\end{aligned} \tag{2}
$$

where the superscript $O(\epsilon)$ represents the first-order corrected quantity, and the correction to the partition function is given as

$$
Z^{O(\epsilon)} = Z_{free}\Bigg(1 + i\epsilon\int_0^{i\beta} d\tau \, \langle X^4(\tau)\rangle_\beta\Bigg) \tag{3}
$$

Before the proof, let's make some notes:

- Each argument in each integral is a Wightman correlator. The terms can all be found straightforwardly from Wick's Theorem, and each of these integrals is solveable. Thus, we can implement an algorithm in Mathematica to solve these.
- The formula as given is completely agnostic of the orderings of the times t_i. i.e. It applies no matter what the OTO-folding number of the contour is.

Proof. The main idea is to parameterize the time contour $C = \{0 \to t_1, t_1 \to t_2, ..., t_{n-1} \to t_n, t_n \to i\beta\}$ by some variable α, and represent the time contour as $t(\alpha)$. Then, we can represent the action of some curve $x_\mu(\alpha) = (x(\alpha), t(\alpha))$ in space-time as (assuming α ranges from $0 \to \alpha_f$)

$$
S[x_\mu] = \int_0^{\alpha_f} L(x, \frac{dx}{d\alpha}\frac{d\alpha}{dt})\frac{dt}{d\alpha}d\alpha \tag{4}
$$

since $dt = \frac{dt}{d\alpha}d\alpha$ and $\frac{dx}{d\alpha}\frac{d\alpha}{dt} = \frac{dx}{dt}$. Then, let α_i be the values of α so that $t(\alpha_i)$ are the points on the contour where we insert the operators. We can represent the expectation value in terms of the path integral

$$\langle X(t_1)...X(t_n)\rangle_\beta = \frac{1}{Z}\int [Dx_\mu] x(\alpha_1)...x(\alpha_n) e^{iS[x_\mu]} \tag{5}$$

This representation of the path integral gives the correct operator ordering due to what positions α_i represent on the contour. To prove the lemma, we must show two things. First, that the perturbed partition function is of the form (3). This is clear by standard path integral derivations. The next step is also standard, expanding

$$e^{iS_{perturbed}[x_\mu]} = e^{iS_{free}}\left(1 + i\epsilon \int_0^{\alpha_f} x^4(\alpha')\frac{dt}{d\alpha'}d\alpha'\right) \tag{6}$$

Then, we can see that

$$\begin{aligned}
&\langle X(t_1)...X(t_n)\rangle_\beta^{O(\epsilon)} \\
&= \frac{1}{Z^{O(\epsilon)}}\left(\int [Dx_\mu] x(\alpha_1)...x(\alpha_n) e^{iS_{free}} + i\epsilon \int_0^{\alpha_f} d\alpha' \frac{dt}{d\alpha'} \int [Dx_\mu] x^4(\alpha') x(\alpha_1)...x(\alpha_n) e^{iS_{free}}\right) \\
&= \frac{Z_{free}}{Z^{O(\epsilon)}}\left(\langle X(t_1)...X(t_n)\rangle_\beta^{free} + i\epsilon \int_0^{\alpha_f} d\alpha' \frac{dt}{d\alpha'} \langle \mathcal{P}_C X^4(t(\alpha')) X(t(\alpha_1))...X(t(\alpha_n))\rangle_\beta^{free}\right)
\end{aligned} \tag{7}$$

where the notation $\mathcal{P}_C$ represents ordering the operators by their position on the contour C. Since the $X(t(\alpha_i))$ in (7) are already ordered by construction, the only factor whose position can vary is $X^4(\alpha')$, whose position depends on the value of α'. Each term in the formula (2) corresponds to each leg of C, and the position of $X^4(\alpha')$ in each term is dependent on the leg. So, we have completed our derivation.

□

There is a generalization for this formula that applies to any order in perturbation theory. I'll explain it for the second order calculation of the two-point function $\langle X(t_1)X(t_2)\rangle_\beta^{O(\epsilon^2)}$ then present the general formula. I will omit the general formula's proof, but hopefully the presentation for $\langle X(t_1)X(t_2)\rangle_\beta^{O(\epsilon^2)}$ is clear enough that it should be apparent how to proceed.

First, let us note that from the same reasoning as before that the second-order correction can be written as follows:

$$
\begin{aligned}
&\langle X(t_1)X(t_2)\rangle_\beta^{O(\epsilon^2)} = \frac{Z_{free}}{Z^{O(\epsilon^2)}}\Bigg(\langle X(t_1)X(t_2)\rangle_\beta^{free} \\
&+ i\epsilon \int_0^{\alpha_f} d\alpha' \frac{dt}{d\alpha'} \langle \mathcal{P}_C X^4(t(\alpha'))X(t(\alpha_1))X(t(\alpha_2))\rangle_\beta^{free} \\
&+ \frac{(i\epsilon)^2}{2!} \int_0^{\alpha_f} d\alpha' \frac{dt}{d\alpha'} \int_0^{\alpha_f} d\alpha'' \frac{dt}{d\alpha''} \langle \mathcal{P}_C X^4(t(\alpha'))X^4(t(\alpha''))X(t(\alpha_1))x(t(\alpha_2))\rangle_\beta^{free} \Bigg)
\end{aligned}
\tag{8}
$$

The first integral in the parentheses is simply the integral for the first-order correction given in formula (2) and can be expanded as such. Let us denote the second integral in (7) by "$correction(O(\epsilon^2))$", without the prefactor of $\frac{Z_{free}}{Z^{O(\epsilon^2)}}$. Then, we can split up "$correction(O(\epsilon^2))$" into nine separate integrals, one for each of the possibilities α', α'' being in the intervals $\{0 \to t_1, t_1 \to t_2, t_2 \to i\beta\}$. So, we if we define each of the intervals of the contour $\{0 \to t_1, t_1 \to t_2, t_2 \to i\beta\}$ as $\{C_1, C_2, C_3\}$ respectively and can define the integrals

$$
I_{ij} = \int_{C_i} dt' \int_{C_j} dt'' \langle \mathcal{P}_C X^4(t')X^4(t'')X(t_1)X(t_2)\rangle_\beta^{free} \tag{9}
$$

Then it's clear that

$$
correction(O(\epsilon^2)) = \frac{(i\epsilon)^2}{2!} \sum_{i,j=1}^{3} I_{ij} \tag{10}
$$

For clarity, let us explicitly write some of these I_{ij} without using the path-ordered notation $\mathcal{P}_C$. For example

$$
\begin{aligned}
I_{11} &= \int_0^{t_1} dt' \int_{t'}^{t_1} dt'' \, \langle X^4(t') X^4(t'') X(t_1) X(t_2) \rangle_\beta^{free} \\
&\quad + \int_0^{t_1} dt' \int_0^{t'} dt'' \, \langle X^4(t'') X^4(t') X(t_1) X(t_2) \rangle_\beta^{free} \\
&= 2 \int_0^{t_1} dt' \int_{t'}^{t_1} dt'' \, \langle X^4(t') X^4(t'') X(t_1) X(t_2) \rangle_\beta^{free} . \\
I_{22} &= 2 \int_{t_1}^{t_2} dt' \int_{t'}^{t_2} dt'' \, \langle X(t_1) X^4(t') X^4(t'') X(t_2) \rangle_\beta^{free} . \\
I_{12} &= \int_0^{t_1} dt' \int_{t_1}^{t_2} dt'' \, \langle X^4(t') X(t_1) X^4(t'') X(t_2) \rangle_\beta^{free} . \\
I_{21} &= \int_{t_1}^{t_2} dt' \int_0^{t_1} dt'' \, \langle X^4(t'') X(t_1) X^4(t') X(t_2) \rangle_\beta^{free} . \\
I_{13} &= \int_0^{t_1} dt' \int_{t_2}^{i\beta} dt'' \, \langle X^4(t') X(t_1) X(t_2) X^4(t'') \rangle_\beta^{free} .
\end{aligned} \tag{11}
$$

Let's say a few things about these formulas:

- For I_{11}, we have to split the ordering up into two separate integrals, each of which has a different operator ordering. However, each of these terms is the same by symmetry of the integration bounds, so we can express the full value as twice the first term. The same property holds for I_{22} and I_{33}.
- Note that $I_{12} = I_{21}$. To see this, we note that switching the order of integration gives the exact same result. For an explanation of why we would expect this, note that if t' and t'' are on separate legs of the contour, $X^4(t')$ and $X^4(t'')$ will always be path-ordered in the same way relative to each other, so we can use the same operator ordering throughout the integration bounds. It will also be the case that $I_{13} = I_{31}$ and $I_{23} = I_{32}$

So, taking into account these redundancies, the full formula for "*correction*$(O(\epsilon^2))$" can be written as:

$$
\begin{aligned}
&correction(O(\epsilon^2)) = \\
&\qquad (i\epsilon^2)\Big(\int_0^{t_1} dt' \int_{t'}^{t_1} dt'' \, \langle X^4(t')X^4(t'')X(t_1)X(t_2)\rangle_\beta^{free} \\
&\qquad\quad + \int_0^{t_1} dt' \int_{t_1}^{t_2} dt'' \, \langle X^4(t')X(t_1)X^4(t'')X(t_2)\rangle_\beta^{free} \\
&\qquad\quad + \int_0^{t_1} dt' \int_{t_2}^{i\beta} dt'' \, \langle X^4(t')X(t_1)X(t_2)X^4(t'')\rangle_\beta^{free} \\
&\qquad\quad + \int_{t_1}^{t_2} dt' \int_{t'}^{t_2} dt'' \, \langle X(t_1)X^4(t')X^4(t'')X(t_2)\rangle_\beta^{free} \\
&\qquad\quad + \int_{t_1}^{t_2} dt' \int_{t_2}^{i\beta} dt'' \, \langle X(t_1)X^4(t')X(t_2)X^4(t'')\rangle_\beta^{free} \\
&\qquad\quad + \int_{t_2}^{i\beta} dt' \int_{t'}^{i\beta} dt'' \, \langle X(t_1)X(t_2)X^4(t')X^4(t'')\rangle_\beta^{free} \Big)
\end{aligned} \tag{12}
$$

Note that the factor of $\frac{1}{2!}$ in front of $i\epsilon^2$ dropped out. This is not an accident, and in fact will be a feature of the general all orders formula, and arises from the combinatorics of the redundant information in the integrals. We are now in a position to state the general formula up to any order.

Lemma 2. *The kth-order correction correction to the n-point function $\langle X(t_1)...X(t_n)\rangle_\beta$ is given by*

$$
\langle X(t_1)...X(t_n)\rangle_\beta^{O(\epsilon^k)} = \frac{Z_{free}}{Z^{O(\epsilon^k)}}\Big(\langle X(t_1)...X(t_n)\rangle_\beta^{free} + \sum_{m=1}^{k} correction(O(\epsilon^m))\Big) \tag{13}
$$

where $Z^{O(\epsilon^n)}$ is given below, and

$$
correction(O(\epsilon^m)) = (i\epsilon)^m \sum_{1\le i_1\le ...\le i_k\le n+1} J_{i_1,i_2,...,i_k} \tag{14}
$$

and

$$
J_{i_1,i_2,...,i_k} = \int_{l_{i_1}}^{u_{i_1}} dt^{(1)} \int_{l_{i_2}}^{u_{i_2}} dt^{(2)} ... \int_{l_{i_k}}^{u_{i_k}} dt^{(k)} \, \langle \mathcal{P}_C X^4(t^{(1)})...X^4(t^{(k)})X(t_1)...X(t_n)\rangle_\beta^{free} \tag{15}
$$

The $t^{(m)}$ are dummy variables of integration. The respective bounds of integration u_{i_m} and l_{i_m} are given and explained in words below. The ordering of the operators implicitly given *in the expression* $\langle \mathcal{P}_C X^4(t^{(1)})...X^4(t^{(k)})X(t_1)...X(t_n)\rangle_\beta^{free}$ *is the same within each integral due to the bounds we put on the integral, and can be found be inserting the* $X^4(t^{(m)})$ *in the correct order in between the* $X(t_i)$ *corresponding to the values of* $i_1, i_2, ..., i_k$.

$$u_{i_m} = \begin{cases} t_{i_m}, \text{ if } \ i_m < n+1 \\ i\beta, \text{ if } \ i_m = n+1 \end{cases} \qquad l_{i_m} = \begin{cases} 0, \text{ if } \ i_m = 1 \\ t_{i_{m-1}}, \text{ if } \ i_m \neq 1 \text{ and } \ i_m \neq i_{m-1} \\ t^{(m-1)}, \text{ if } \ i_m \neq 1 \text{ and } \ i_m = i_{m-1} \end{cases} \tag{16}$$

(16) means that the upper bounds of integration for the m-th dummy variables are given by the upper bound of one of the (n+1) legs of contour $C = \{0 \to t_1, t_1 \to t_2, ..., t_n \to i\beta\}$. *A lower bound is given by the lower bounds of the leg if the m-th dummy variable is the first variable in its leg, or it's given by the previous dummy variable,* $t^{(m-1)}$ *if it's not the first in the leg. This prescription for the integration bounds makes sure the term is properly time-ordered.*

The formula for $Z^{O(\epsilon^k)}$ *is below, and the reason for the cancellation of the* $m!$ *factors arising from the exponential is again due to degeneracy of orderings.*

$$Z^{O(\epsilon^k)} = Z_{free} + \sum_{m=1}^{k} (i\epsilon)^m \int_0^{i\beta} dt^{(1)} \int_{t^{(1)}}^{i\beta} dt^{(2)} ... \int_{t^{(m-1)}}^{i\beta} dt^{(m)} \, \langle X^4(t^{(1)})...X^4(t^{(m)})\rangle_\beta^{free} \tag{17}$$

I will now list some final comments:

- The formulas bear a resemblance to the Dyson series for the time-ordered exponential.
- The formulas given are implemented in a Mathematica notebook. To do these calculations, we can apply Wick's theorem and generate contractions and diagrams for each term that are then integrated. The notebook takes care of the contractions and shows the diagrams and their multiplicities, including disconnected ones.
- To save additional computational time, we can WLOG shift all of the times in these Wightman correlators down by t_1, i.e. $t_1 \to 0, t_2 \to t_2 - t_1, t_3 \to t_3 - t_1, ...$, but keeping the last upper bound $i\beta$ the same. This gets rid of integrals whose outermost lower bound is zero and gives the same results due to time translational invariance. Or equivalently, we can keep all the times fixed and move $i\beta \to i\beta + t_1$, but drop integrals whose outermost lower bound is zero. This alternative amounts to making the starting time of the paths t_1 instead of zero.
- While these formulas are not expressed in the language of Schwinger-Keldysh, I suspect they give the same results as the Schwinger-Keldysh diagrammatics. This would not be surprising, since these calculations are done using standard perturbative expansions.

- Secular terms remain an issue in these expansions.

3 References

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