Computer Simulations of a Superfluid Vortex with a Non-trivial Boundary

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This paper summarizes the work I did on superfluid vortex simulations based on previous results in Professor Rena Zieve's Group at UC Davis during the summer of 2016. It reports the detailed process of simulating the motion of a superfluid vortex in a non-trivial geometry, which is realized by incorporating a publicly available Laplace solver into a separate simulation code. The boundary correction proves to be the possible reason for the vortex's more downward precession near the transition. In addition, a wiggling signal that shows up near the local minimum of that period is suspected of being caused by the boundary too.

INTRODUCTION

In 1937, superfluid effect was found by Pyotr Kapitsa [1] and John F. Allen, and Don Misener [2] in liquid ^4He. According to the phase diagram of helium-4 FIG. 1, normal fluid (He-I) turns into superfluid (He-II) through a phase transition at a low temperature (e.g. at 2.17K under saturated vapor pressure). In analogy with a superconductor, which has no resistance for electrical current, superfluid has no resistance for fluid flow. Superfluid is also incompressible and has incredibly high thermal conductivity [3] as well as its zero viscosity. More intriguingly, the vortex in superfluid ^4He was found to be quantized by Hall and Vinen in 1956 [4]. Microscopically, this property could be explained by a generalized Bose Einstein condensation. A single-particle wavefunction can thus be used to describe the superfluid helium-4 system [5]

$$\Psi(r) = A(r)e^{i\Theta(r)} \tag{1}$$

where Θ represents the phase of the superfluid. The velocity of the superfluid is defined as

$$\mathbf{v} = \frac{\hbar}{m_{He}}\nabla\Theta \tag{2}$$

Therefore, the circulation in the superfluid, which is given by a closed loop integral of the velocity, equals zero, if Stokes' theorem can be applied.

$$\begin{aligned}\kappa &= \oint_{loop} \mathbf{v}\cdot d\mathbf{l} \\ &= \frac{\hbar}{m_{He}}\oint_{loop} \nabla\Theta\cdot d\mathbf{l} \\ &= \frac{\hbar}{m_{He}}\int_{surface} \nabla\times(\nabla\Theta)dA \text{ (Stokes' Theorem)} \\ &= 0\end{aligned} \tag{3}$$

In other words, superfluid is irrotational

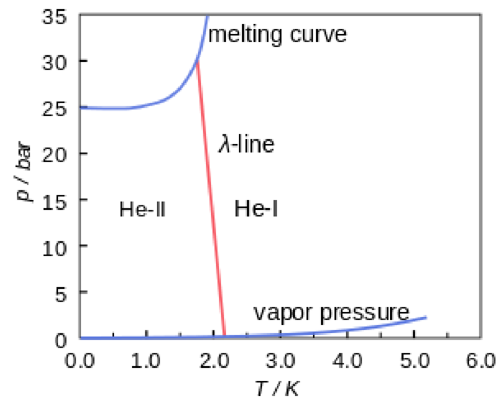

FIG. 1. Phase diagram of ^4He

$$\nabla\times\mathbf{v} = 0 \tag{4}$$

unless the region of superfluid is not simply connected so that Stokes' theorem no longer applies. In this case, the circulation can be an integer times a constant, since the closed loop integral of $\nabla\Theta$ gives $2\pi n$.

$$k = \frac{\hbar}{m_{He}}\oint_{loop} \nabla\Theta\cdot d\mathbf{l} = \frac{nh}{m_{He}} \tag{5}$$

In other words, superfluid vortices are quantized. The source of these quantized vortices is called vortex core or vortex for simplicity, whose existence prevents the region of superfluid being simply connected.

The novel properties of superfluids, including zero viscosity, incompressibility and especially quantized vortices, enormously simplify the study of vortices in hydrodynamics. In Professor Rena Zieve's Group at the University of California, Davis, experiments and corresponding computer simulations are conducted to study the motions and interactions of vortices in superfluid ^4He which provide insights into vortex dynamics in normal

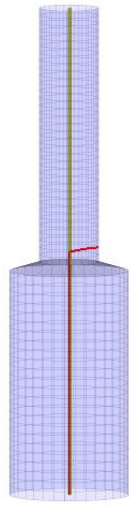

FIG. 2. Schematic of the double-diameter cell. The yellow line represents a metal wire and the red is a vortex core.

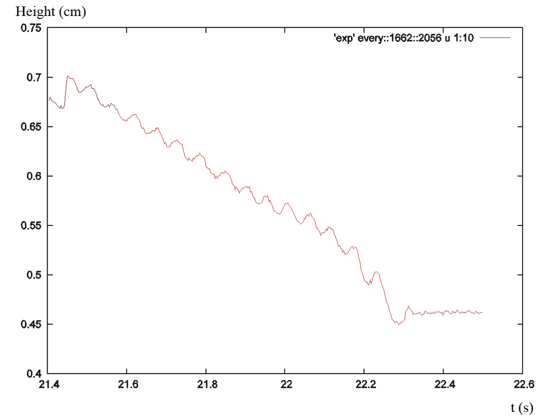

FIG. 3. Experimental data of the heights of the attachment point

fluids, in type II superconductors, as well as the rotational properties of neutron stars. My part of the work during the summer of 2016 in Professor Zieve's group was to consider and analyze the contribution of a nontrivial boundary when simulating the motion of a superfluid vortex over time by comparing the simulations with the corresponding experimental results.

TECHNICAL BACKGROUND

Outline of the Experimental Setup

In the experiment, a refrigerator is used to cool down the ^4He in a cell that consists of two different cylinders connected by a circular truncated cone. As FIG. 2 shows, a thin metal wire (16 μm diameter) runs along the middle of the cell and provides a low-energy spot that vortex core prefers to stay at. It is important to note that the system is not only translationally but also rotationally asymmetrical since the wire is slightly off-center. In some common non-equilibrium cases, part of the vortex gets excited, decays off the wire and stretches to the cell wall. This off-wire part is called a free vortex, since it is free to move in the superfluid. The point where a free vortex attaches to the wire is called the attachment point.

What is more, a magnetic field is added perpendicular to the metal wire that carries an electric current. The wire vibrates in different modes under the influences of the Lorenz force and the velocity field generated by the vortex. By analyzing the vibrating modes, data on the effective circulation, which is proportional to the length of vortex on the wire (i.e. the height of the attachment point), can be obtained. Furthermore, the motions of a free vortex in the superfluid can be reasonably inferred: The free vortex precesses around the off-center wire. It moves downward because its energy dissipates at a certain rate, which is the general slope of the curve in FIG. 3. In addition, it does not only move down on average, but also goes up and down periodically, because the wire is not on-center. The attachment point goes up because the extra free vortex goes back on the wire when the free vortex rotates from the far side to the close side, and vice versa.

Besides inferring the motions of a vortex from vibrating modes of the wire, computer simulations can help us gain more insights into how a free vortex moves in the superfluid and confirm or disprove our experimental hypotheses.

Boundary Conditions

In analogy to the magnetic field, the superfluid velocity field also has no divergence and no curl except at the vortex core. Therefore, the Biot-Savart law can be used to calculate the velocity generated by the vortex. To use the Biot-Savart law, the vortex should either connect on itself to form a loop or stretch to infinity like currents do. In our case, we add image vortices, which extend to infinity, to the two ends of the vortex in the cell. The velocity field generated by the vortex can be calculated by the Biot-Savart law, which requires an integral from one infinity to the other.

Magnetic Field	Superfluid Velocity Field
$\nabla \cdot \vec{B} = 0$	$\nabla \cdot \vec{v} = 0$
$\nabla \times \vec{B} = 0$(except current)	$\nabla \times \vec{v} = 0$(except vortex)
$\vec{B} = \frac{\mu_0}{4\pi} \int_L \frac{I d\vec{l} \times \vec{r}'}{\lvert \vec{r}' \rvert^3}$	$\vec{v} = \frac{\kappa}{4\pi} \int_L \frac{d\vec{l} \times \vec{r}'}{\lvert \vec{r}' \rvert^3}$

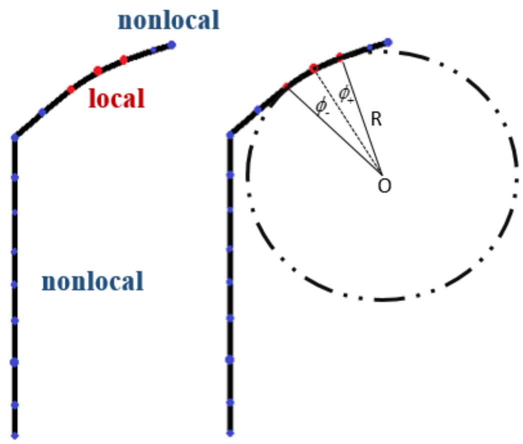

FIG. 4. Schematic of the calculation method for the local vortex

As mentioned before, since the container is translationally and rotationally asymmetric, even the velocities generated by a straight vortex along the wire are not entirely tangent to the surface. The error of the velocity caused by not fitting the boundary conditions will be bigger when there is a free vortex segment. In order to solve the boundary problem, it is important to note that the superfluid velocity satisfies Laplace's equation, since it has no curl nor source.

$$\nabla^2 \vec{v}_b = 0 \tag{6}$$

Set the velocities at the boundary to be the projections of the velocities generated by the vortex on the normal vectors of the boundary points.

$$v_{x,b} = v_{\perp x} = (\hat{x} \cdot \hat{n}) v_x \tag{7}$$
$$v_{y,b} = v_{\perp y} = (\hat{y} \cdot \hat{n}) v_y \tag{8}$$
$$v_{z,b} = v_{\perp z} = (\hat{z} \cdot \hat{n}) v_z \tag{9}$$

Technically, as the flowchart FIG. 5 shows, we select nodes in and on our particular cell, calculate the velocities on the nodes with Biot-Savart law, project the velocities at the nodes onto the normal vectors on the boundary, create a triangle mesh in preparation for the interpolation, and send the nodes, mesh and perpendicular velocities to dep Solver, which is publicly available software using linear interpolation and Indirect Formulation of the Boundary Element Method to solve Laplace's Equation. Finally, by subtracting the boundary solution from the velocity generated by the Biot-Savart law, the velocity field satisfying the boundary condition can be obtained.

Simulation of Motion

To simulate the motion of a superfluid vortex over time, Professor Zieve's group wrote a code based on superfluid vortex dynamics [6]. To put it simply, when calculating the velocity of a point on the vortex, the Biot-Savart law no longer converges as the denominator goes to zero. However, knowing the propagation velocity of a quantized vortex ring,

$$\dot{\vec{s}}(\text{ring}) = \frac{\kappa}{4\pi} \frac{\hat{z}}{R} ln(\frac{8R}{e^{1/4} a_0}) \tag{10}$$

we could draw a ring which passes through the point we calculate the velocity of and the two points next to it FIG. 4. By using the Biot-Savart law to integrate the velocity along the major arc and subtracting the integral from the velocity of the ring, we can get the local contribution of the velocity.

$$\dot{\vec{s}}(\text{arc}) = \frac{\kappa}{4\pi} \frac{\hat{z}}{R} ln[\cot(\frac{1}{4} \mid \phi_+ \mid) \cot(\frac{1}{4} \mid \phi_- \mid)] \tag{11}$$
$$\begin{aligned} \dot{\vec{s}}_{local} &= \dot{\vec{s}}(\text{ring}) - \dot{\vec{s}}(\text{arc}) \\ &= \frac{\kappa}{4\pi} \frac{\hat{z}}{R} ln[\frac{2(l_+ l_-)^{1/2}}{e^{1/4} a_0}] \end{aligned} \tag{12}$$

In addition, since the elementary excitation of superfluid ^4He free vortex is strongly scattered by quantized vortices, there is a frictional force depending on the relative velocity between the gas of elementary excitations and the vortex. This frictional force will be exerted on the fluid near the core and will give the free vortex an additional velocity.

$$\dot{\vec{s}}_f = \alpha \vec{s}' \times (\vec{v}_n - \dot{\vec{s}}_0) - \alpha' \vec{s}' \times [\vec{s}' \times (\vec{v}_n - \dot{\vec{s}}_0)] \tag{13}$$

In conclusion, the velocity of the point on the free vortex is the sum of the local contribution, nonlocal contribution, the velocity driven by frictional force minus the boundary correction, which is neglected in previous cases since it has comparably small contributions in a container with high symmetry.

$$\dot{\vec{s}} = \dot{\vec{s}}_{local} + \dot{\vec{s}}_{nonlocal} + \dot{\vec{s}}_f - \dot{\vec{s}}_b \tag{14}$$

As the vortex moves, the velocities of the points on the vortex change accordingly. Using Runge-Kutta-Fehlberg method as a time-stepper, the motion of a superfluid vortex over time can be simulated.

RESULTS AND DISCUSSION

To test the Laplace solver, I set the initial velocity field to be a constant flow and solved the boundary problem with the solver. As the bottom view shows in FIG. 6, there is no circulation nor source in it. At the edge, it satisfies the boundary conditions. In the transition, the

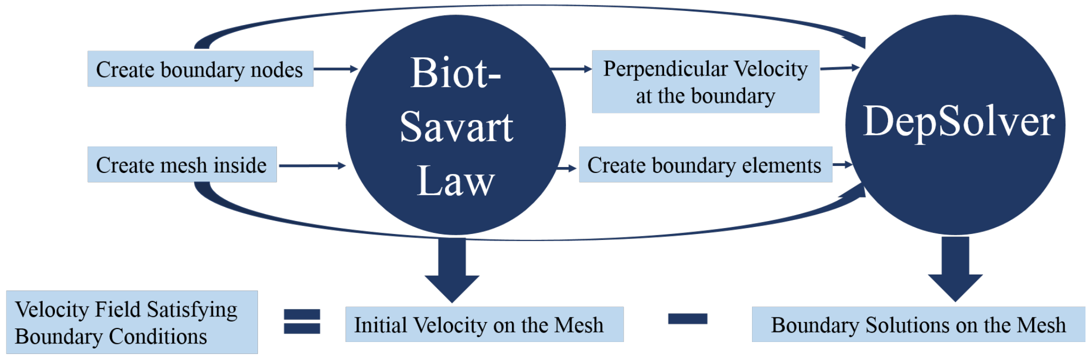

FIG. 5. A flowchart on how to solve the boundary condition in the particular geometry

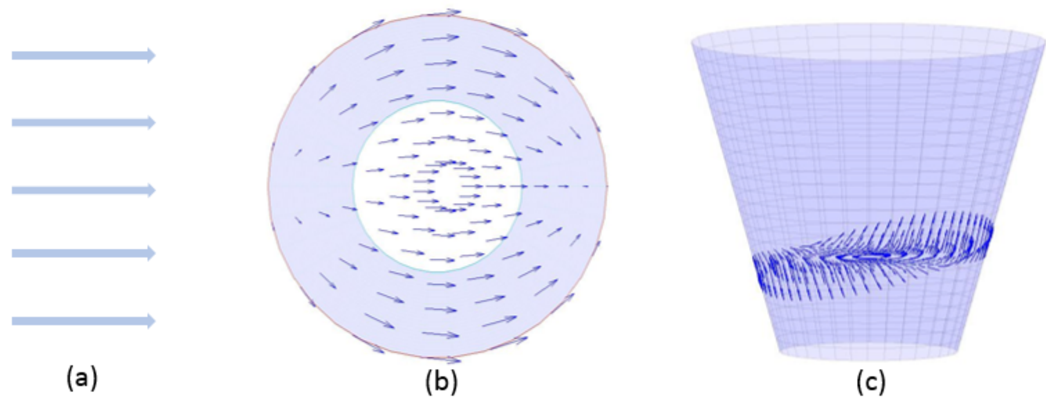

FIG. 6. (a) Constant test flow (b) Bottom view of the cell (c) velocity field in the transition

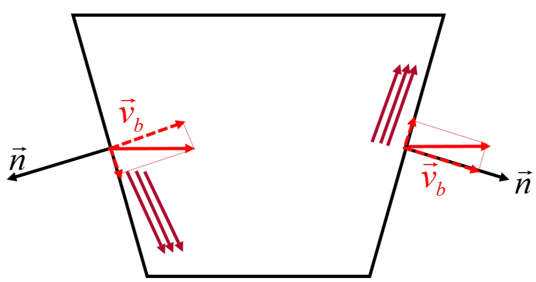

FIG. 7. Schematic of the reason for the asymmetric field when fluid flows constantly

velocity field also nicely satisfies the boundary conditions. It is worth noting that there is an asymmetric upward velocity field in one half and a downward in the other half. Intuitively, when the constant flow hits into the transition, its perpendicular component will be cancelled out by solving the boundary problem FIG. 7. Therefore, a downward velocity field is left when the flow hits into the transition and an upward velocity field is left when the flow flows out of the transition. A similar phenomenon would happen when there is an off-center straight vortex in the cell. The fluid flows around the vortex core, and because it is off-center, it will hit the wall. As FIG. 8 shows, the left arrow hits into the transition, and the right arrow hits out of the transition. In this case, there will also be a downward velocity field in the one half and an upward in the other. Essentially, the velocity field inside the cell must not be simply circular since the off-center vortex breaks the rotational symmetry of the system.

Experimentally, we always see a much more downward procession when the free vortex is near the transition. However, before involving the boundary correction in, we cannot see this behavior in our simulation FIG. 9. With boundary problem solved, which is represented by the red line, we could see for a period far from the transition, the boundary correction makes free vortex move slightly upwards than usual FIG. 10. For a period near the transition, the free vortex moves much more downwards with the boundary correction, which indicates that the correction from boundary solution might be the reason for the more downward procession near the transition. In addition, a wiggling signal, which shows up near the local minimum of the period, also happens in the experiment. However, judgement on whether these two wiggling signals show up for the same reason needs more confirmation.

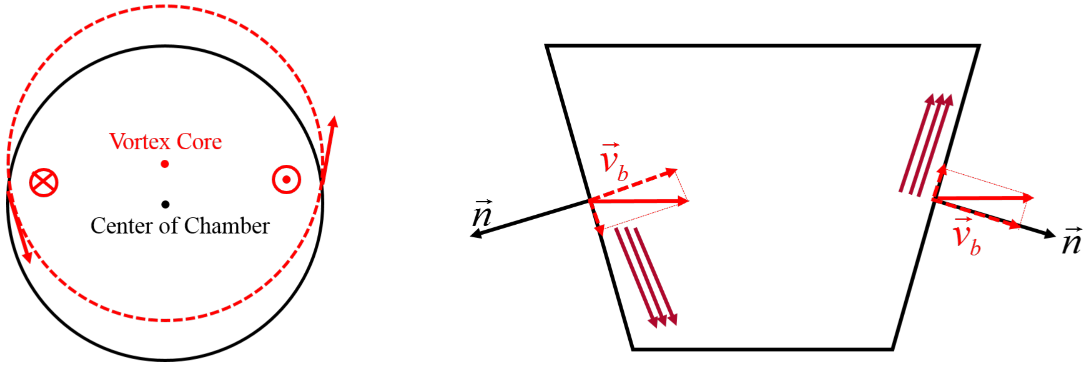

FIG. 8. Schematic of the reason for the asymmetric field when there is an off-cente vortex

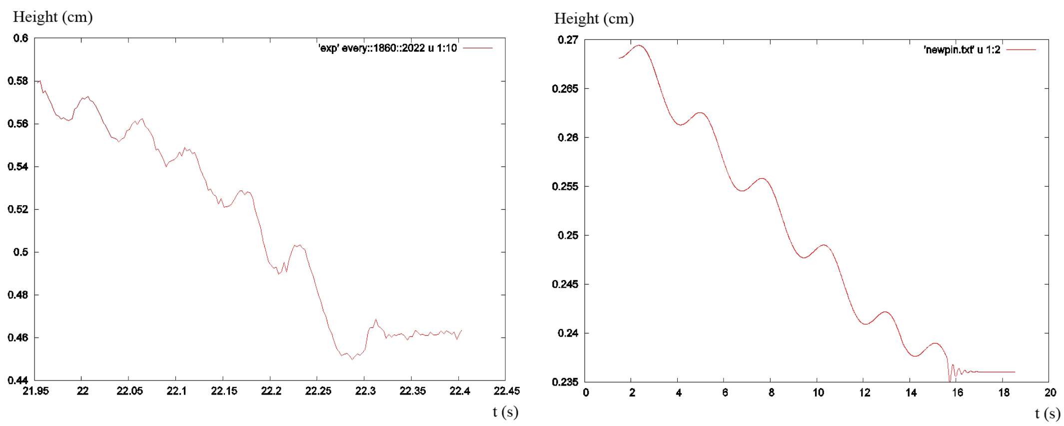

FIG. 9. Comparison between experimental data of the height of attachment point (left) and the numerical simulation without boundary corrections (right)

CONCLUSION AND FUTURE WORK

In conclusion, this paper reports the detailed process of simulating the motion of a superfluid vortex in a non-trivial geometry, which helps develop further insights into the corresponding experiment results. The boundary correction proves to be the possible reason for the vortex's much more downward precession near the transition. In addition, a wiggling signal that shows up near the local minimum of that period is suspected of being caused by the boundary too.

In the future, the program should be optimized to lower the time cost of running it. Optimization methods include using more efficient interpolation methods when solving the Laplace's equation, reducing the input and output in the cooperation between the simulation code and the Laplace solver, and trying to find feasible methods from first principles to avoid solving Laplace equation every hundred steps. Once the program is optimized, run the program with real parameters through an entire process, and compare the simulations with the experimental data to obtain more reliable discoveries and conclusions. In addition, simulations in different geometries can be tried to further explore the superfluid vortex dynamics in complex conditions.

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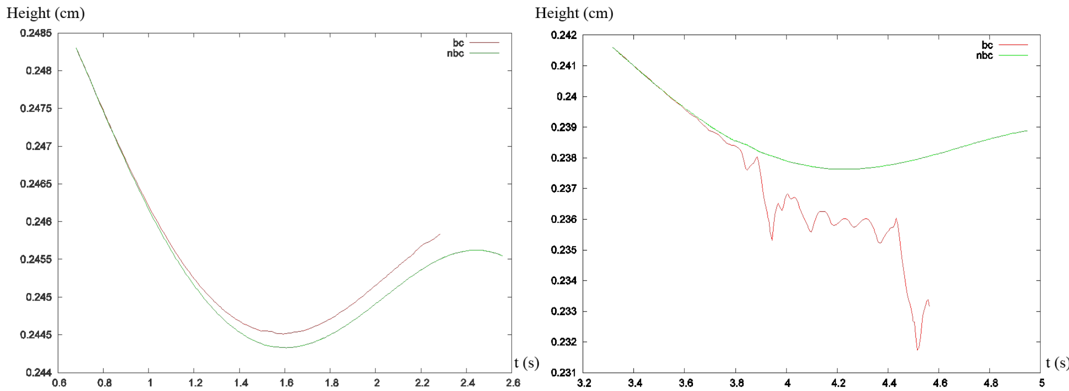

FIG. 10. Simulations with boundary corrections for a period far from the transition (left) and a period near the transition (right)