Temperature Dependence of Entanglement Negativity in Lattice Models: Area Laws and Sudden Death

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We use exact diagonalization (ED) and numerical linked cluster (NLC) expansion to study bipartite entanglement negativity (EN) as a function of temperature in quantum lattice models. In the T=0 limit, we check our results against ground state calculations for the Renyi entropy with index $\frac{1}{2}$. Comparison between ED and NLC in 1D gives us confidence in the NLC method, which is then applied to 2D also. We study transverse-field Ising model (TFIM) and XXZ models. NLC shows good convergence as long as the system does not have long-range correlations. In all cases studied, we find that EN satisfies area-law and has sudden death which survives in the thermodynamic limit. EN generally decreases monotonically with increase in temperature. The low negativity at high-T is in sharp contrast to bipartite von Neumann entropy, which is generically proportional to the volume and approaches the thermal entropy. The negativity measure is also quite different from von Neumann entropy based mutual information (MI) at finite temperatures and does not have signatures of classical phase transitions. We argue that low true quantum-entanglement in the highly mixed thermal states reflects the fact that the thermal state arises from strong entanglement with the environment, which leaves only limited room for entanglement between different parts of the subsystem.

INTRODUCTION

Quantum entanglement is arguably the most mysterious phenomenon that falls out of the theory of Quantum Mechanics. When studying a one particle system, a system can be in a superposition of states, as if it is in multiple states at the same time. When a second particle is added we have product states, where to describe the state of the system as a whole, the state of each particle must be specified. Now, when studying a two particle system, a system can be in a superposition of product states. Consider a system of two particles, where the wave function is known to be a superposition of product states. The strangeness occurs upon making a measurement; if the first particle is measured, the wave function collapses to this particular product state, and the observer now instantly has information about the second particle, this is known as quantum entanglement.

Some of the first people to study quantum entanglement were Einstein, along with Podolsky and Rosen [1]. Their general feeling was that quantum entanglement implied that the theory of Quantum Mechanics was incomplete, as they could not imagine a world in which this phenomenon existed. This was a very controversial topic in Physics for decades, until nearly 50 years later, Alain Aspect devised an experiment that finally proved the existence of quantum entanglement [2].

In recent years, the thermodynamic properties of quantum entanglement have become a topic of great interest. Interesting questions arise such as, can entanglement be seen on a macroscopic level? How does entanglement formation depend on time? temperature? distance? system size? In the process of trying to answer these questions, it was realized that entanglement is a rather good probe into some of the deepest questions in Condensed Matter Physics, such as when quantum statistical mechanics can be used on an isolated quantum system, or studying quantum phase transitions.

Entanglement is a very difficult quantity to measure in the lab, and is also rather difficult to quantify theoretically. Due to the probabilistic nature of Quantum Mechanics, any state can be expressed as a superposition of states, where the superposition coefficients relate to the probability of being in a certain state. If one thinks of these probabilities as defining a probability distribution, then one can associate an entropy to the state of interest. If the basis chosen is the Schmidt basis, obtained from the generalized Gram-Schmidt orthonormalization procedure to many body systems, then the associated entropy is named the von Neumann entropy, and this quantity is intimately related to the entanglement. Much of modern research in quantum entanglement is done via the quantity of von Neumann entropy, and other related quantities such as Renyi entropies. The issue with these quantities is realized when considering a mixed state, where a system can not be described entirely by a wave function, but is rather described by a density matrix. Mixed states arise when the quantum mechanical state of a system is known only with some classical probability. One example is when considering a thermal state, it is known that the state of the system will be in an energy eigenstate, but the probability of being in any given eigenstate is given by the classical Boltzmann factor. When a mixed state is being studied, there is entropy due simply to the fact that the system is mixed. This leads to entropy based measures of entanglement having a difficult time distinguishing the entropy of entanglement from the entropy of being mixed. Many measures of entanglement in a mixed state have been developed in quantum information the-

ory, but most of them are difficult to compute even for relatively small quantum systems. The measure of (logarithmic) negativity was proposed by Vidal and Werner [3] as an effective, computable measure of entanglement in a mixed state.

In this study we focus primarily on the thermal nature of entanglement negativity. Since the entropy based measures of entanglement are flooded with classical correlations when studying mixed states, negativity provides a useful probe for studying entanglement in the thermal state. Here we provide a computational study of entanglement negativity for one-dimensional chains and the two-dimensional square lattice. For more details pertaining to this study, see [8].

MODELS AND METHODS

If working with a bi-partitioned system, with one partition labeled A, and the other B, then (logarithmic) negativity is defined in the following way

$$\mathcal{N} = \ln ||\rho^{\Gamma_B}||_1 \tag{1}$$

where ρ^{Γ_B} denotes the partial transpose of the density matrix with respect to subsystem B, and $||X||_1$ denotes the trace norm of X.

If we have a density matrix acting on the Hilbert space $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$, call it ρ, then the matrix elements of ρ^{Γ_B}, when working in an orthonormal basis $\{|A_n, B_m\rangle\}$, are obtained in the following manner

$$\langle A_i, B_j| \rho^{\Gamma_B} |A_k, B_l\rangle \equiv \langle A_i, B_l| \rho |A_k, B_j\rangle$$

Where indices B_j and B_l are swapped.

If considering a pure state, it is shown in [3] the following is true

$$||\rho^{\Gamma_B}||_1 = \left[\sum_\alpha C_\alpha\right]^2 \tag{2}$$

where $\{C_\alpha\}$ are the Schmidt coefficients. If we consider the reduced density matrix ρ_A for a pure state $|\Psi\rangle$, obtained by tracing out B degrees of freedom from the full density matrix

$$\rho_A = \underset{B}{\mathrm{Tr}}(\rho) = \underset{B}{\mathrm{Tr}}(|\Psi\rangle\langle\Psi|)$$

if working in the Schmidt basis, we can write $|\Psi\rangle = \sum_\gamma C_\gamma |A_\gamma\rangle |B_\gamma\rangle$, And in this basis the reduced density matrix ρ_A is

$$\rho_A = \underset{B}{\mathrm{Tr}}(|\Psi\rangle\langle\Psi|) = \sum_\alpha \langle B_\alpha| (|\Psi\rangle\langle\Psi|) |B_\alpha\rangle$$

$$= \sum_{\alpha\gamma\beta} C_\gamma C_\beta \langle B_\alpha|A_\gamma, B_\gamma\rangle \langle A_\beta, B_\beta|B_\alpha\rangle$$

$$= \sum_{\alpha\gamma\beta} C_\gamma C_\beta |A_\gamma\rangle\langle A_\beta| \delta_{\alpha,\gamma}\delta_{\alpha,\beta}$$

$$\implies \rho_A = \sum_\alpha C_\alpha^2 |A_\alpha\rangle\langle A_\alpha| \tag{3}$$

implying that the eigenvalues of ρ_A are the squares of the Schmidt coefficients. So to compute negativity for a pure state, we constructed and numerically diagonalized ρ_A and utilized equations (1, 2 & 3). Another measure of entanglement gathered from ρ_A are the well studied Renyi entropies of order α defined as

$$S_\alpha = \frac{1}{1-\alpha} \ln\left[\mathrm{Tr}\left(\rho_A^\alpha\right)\right]$$

with the tactic used for computing the negativity, it is simple to see that the negativity is equal to a Renyi entropy of index $\frac{1}{2}$.

When a mixed state is being studied, i.e. a thermal state, equation (2) can not be utilized, and the laborious partial transposition and full diagonalization must be performed to compute the negativity. Since the thermal properties of negativity are what is of most interest in this study, many times these laborious tasks can not be circumvented. Even when considering thermal properties, calculations on systems with 12 or 14 sites, for one temperature value, can be computed in about one day on a single personal computer, where when studying the ground state, systems with 24 or 26 sites can be studied in the same time using a Lanczos algorithm.

When considering the thermal properties, a thermal density matrix was constructed defined as

$$\rho = \frac{e^{-\beta\mathcal{H}}}{\mathcal{Z}}$$

where $\mathcal{Z}$ is the partition function, β is reciprocal temperature, and $\mathcal{H}$ is the Hamiltonian for the system of interest. For this study, two Hamiltonians were considered using lattice models. First, an antiferromagnetic XXZ model Hamiltonian was used defined as

$$\mathcal{H} = \sum_{\langle i,j\rangle} (S_i^z S_j^z + \lambda[S_i^x S_j^x + S_i^y S_j^y])$$

where the summation is taken over nearest neighbors. When $\lambda < 1$ we are in the Ising regime, $\lambda > 1$ is the XY regime, and at $\lambda = 1$ perfect Heisenberg symmetry is realized. This Hamiltonian is block diagonal in the basis of eigenstates of S_{tot}^z, where all elements in a particular block have the same magnetization. Also the transverse-field Ising model (TFIM) was studied with Hamiltonian

$$\mathcal{H} = -J \sum_{\langle i,j\rangle} S_i^z S_j^z - h \sum_i S_i^x$$

with the first sum over nearest neighbors, and the second sum over all sites. For this study, h was set to unity, effectively setting our energy scale.

To reduce finite-size effects two approaches were used, namely applying periodic boundary conditions (PBC) and employing a Numerical Linked Cluster (NLC) expansion. The method of applying PBC is to simply bond the last site in a chain to the first, effectively forming a ring. As for NLC, a brief explanation will be presented, and for a more elaborate discussion I point the reader to [4, 5]. This method is based on defining some property of a given system of interest in the following way

$$\mathcal{P}(\mathcal{O}) = \sum_{c \subseteq \mathcal{O}} \mathcal{W}(c)$$

with $\mathcal{O}$ being the order of the calculation, c being a cluster that can be embedded in the system with order $\mathcal{O}$, and $\mathcal{W}(c)$ is the associated weight of cluster c. There is some freedom in picking how to define the order, and in this study it is defined as the number of sites in the lattice. The above relation can be re-expressed to define the weight of a cluster recursively as follows

$$\mathcal{W}(\mathcal{O}) = \mathcal{P}(\mathcal{O}) - \sum_{c \subset \mathcal{O}} \mathcal{W}(c)$$

with the sum now running over the proper subsets only, and the base cases for this recursive definition are that $\mathcal{P}(2) = \mathcal{W}(2)$ and both the property and weight for a system of only one site or zero sites is 0. A partial sum is then computed for all weights up to a given order, where the sum up to order infinite would exactly equal the quantity of interest in the infinite system.

Since in a pure state, negativity is equivalent to a Renyi entropy of index $\frac{1}{2}$ it should satisfy the field theory results for ground state Renyi entropies in terms of the central charge, as well as match the same behavior of the entanglement entropies in the ground state. In the ground state, we expected that the negativity should see an area law, with logarithmic corrections needed when the energy spectrum had gapless modes, and this was observed. Also, when examining thermal states, in the limit that $T \to 0$ we should recover the results found in the ground state. However, this breaks down when we have degenerate ground states, such as for odd sites in the XXZ model, in which for any non-zero temperature, no matter how small, the system is in a mixed state, and the negativity is not rightfully equal to a Renyi entropy of index $\frac{1}{2}$. When studying systems with no degenerate ground states, there was great agreement between our study of thermal states in the limit as $T \to 0$, and the ground state values for the negativity.

RESULTS

Ground state

The ground state behavior of the negativity was computed. Being that the ground state is necessarily a pure state, equation (2) was used. To get our hands on the ground state, a Lanczos algorithm was employed. When studying an anti-ferromagnetic system with the XXZ model, only the block of the Hamiltonian corresponding to zero magnetization was used, reducing computational demands. For the XXZ model, it was seen that in the ground state, for values of $\lambda \geq 1$, the negativity scaled logarithmically with the size of the system, and for $\lambda < 1$, the negativity deviates from this logarithmic scaling, and appears to saturate to a constant value. For the TFIM, it was computed that the negativity saturated to a constant for all values of J, except for $J = 2$ in which the negativity diverged logarithmically with increasing system size. These results match those of the famous area law, with logarithmic corrections when there are gapless modes, that is observed in the entanglement entropies, and is observed in Fig. 1. Since in a pure state negativity is a Renyi entropy of order $\frac{1}{2}$ these similarities were anticipated. Negativity should also follow the field theory functional form for the Renyi entropies given by

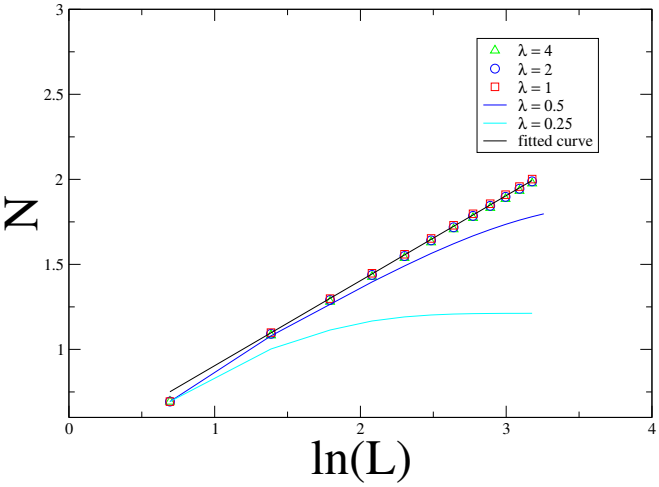

FIG. 1. Negativity for the antiferromagnetic XXZ model as a function of the natural logarithm of length L for periodic chains with varying λ. It is seen that logarithmic divergence is realized for $\lambda \geq 1$.

$$S_\alpha = \frac{c}{6}\left(1 + \frac{1}{\alpha}\right) \ln\left[\frac{2L}{\pi} \sin\left(\frac{\pi x}{L}\right)\right] + \mathcal{O}(1)$$

where c is the central charge, α the renyi index, L the number of sites, and x the number of sites in partition A. Fig. 2 shows the negativity as a function of the ratio $\frac{x}{L}$ for $L = 24$. The data points for $\lambda = 1$ were fitted to a general natural logarithm of a sinusoid, illustrated as the black curve, showing good agreement of our data with this field theory result. The results for the ground state show that the negativity measure encapsulates many of the same qualitative features associated with other measures of entanglement, as it should.

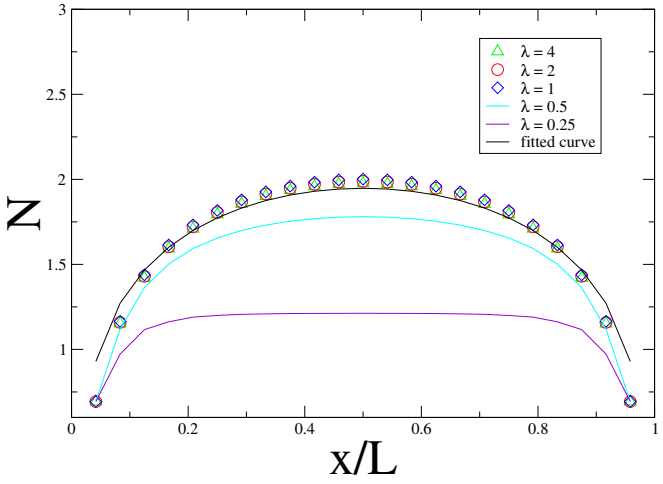

FIG. 2. Negativity for the antiferromagnetic XXZ model as a function of the ratio of size of partition A x to the length L for periodic chains with varying λ. It is seen that logarithmic divergence is realized for $\lambda \geq 1$. The black curve is the data of $\lambda = 1$ fitted to a general sinusoid of a logarithm, showing good agreement with field theory results for a Renyi entropy of index $\frac{1}{2}$.

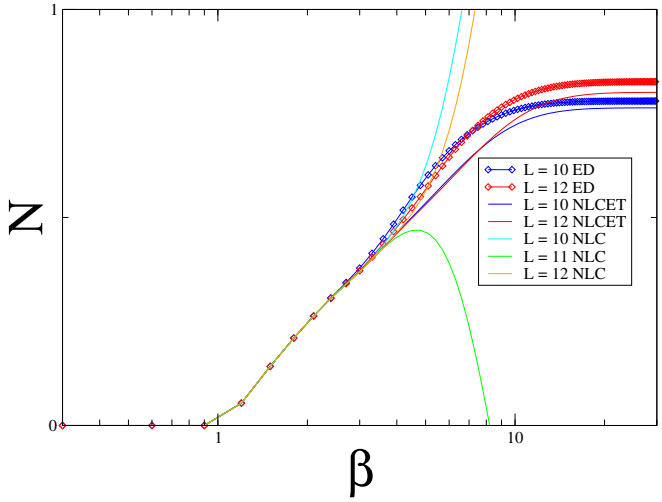

FIG. 3. Logarithmic negativity of the antiferromagnetic Heisenberg model as a function of β calculated by exact diagonalization of rings of size L and by NLC. For the NLC the partial sums of different orders and the Euler sums are shown.

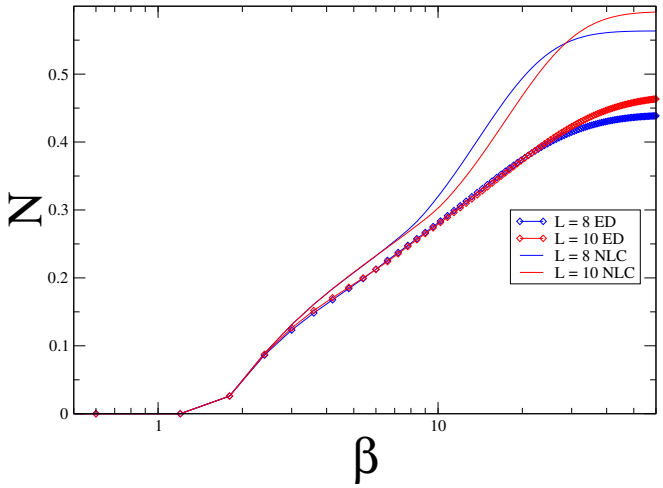

FIG. 4. Logarithmic negativity for the transverse field Ising model (TFIM) tuned to its quantum critical point ($J = 2$) as a function of β calculated by exact diagonalization of rings of size L and by NLC. For the NLC the partial sums of different orders are shown.

Thermal state

The first thermal calculations made were the temperature dependence of negativity for 1D chains and rings of varying length, for both the XXZ model and Ising model for various Hamiltonian parameters. All system variations computed had a few key qualitative features in common. First, there appears to exist a critical temperature that if above, the negativity sees a "sudden death" and vanishes. Next, all system variations showed a monotonic growth of the negativity with increasing β. For the XXZ model, all values of λ computed, it appears that the way in which negativity grows with increasing system size is appreciably different for systems with an even number of sites, oppose to ones with an odd number. This difference between odds and evens is seen in Fig. 3, and was seen to be more prevalent for large β. However, for the Ising model this difference between odds and evens was not as prevalent. For the XXZ model, it appeared that the odds and evens approached different values in the thermodynamic limit, where as for the Ising model, they appeared to be converging to the same value Fig. 4, but for small system sizes there was a non-monotonic oscillatory growth with system size. These differences between odds and evens sometimes yielded alternating series expansions when employing NLC, making it more difficult to determine convergence or divergence.

This sudden death phenomenon was examined in further detail for 1D chains and rings. Since machines are not exactly accurate, and some amount of error is induced purely because of machine precision, some threshold needed to be set to distinguish between a negativity value of zero, from one that is non-zero. This threshold was set initially at 10^{-10}, and the value of β for which this threshold was exceeded was computed using a bisection method. We then varied the threshold value, and observed that for thresholds above 10^{-5} the sudden death values were invariant. Figure 5 illustrates our findings for various values of λ. These data strongly suggest that sudden death survives in the thermodynamic limit.

The thermal properties of the negativity are of most interest, and we wanted to check if the area law held for

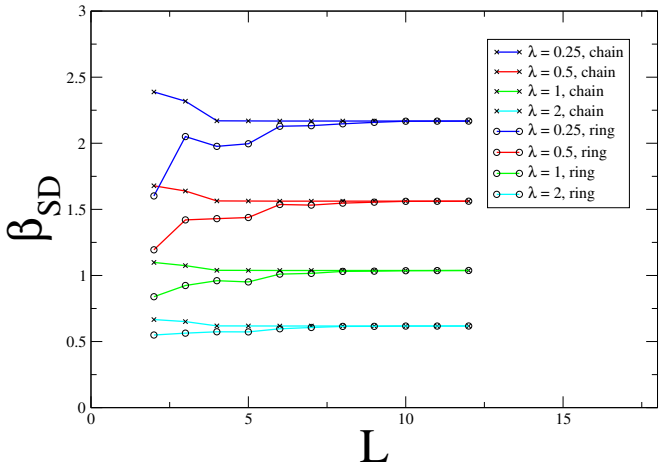

FIG. 5. Sudden death inverse temperature β values for the antiferromagnetic XXZ models for different size rings and chains. Note that the results converge from above for the chains and from below for the rings.

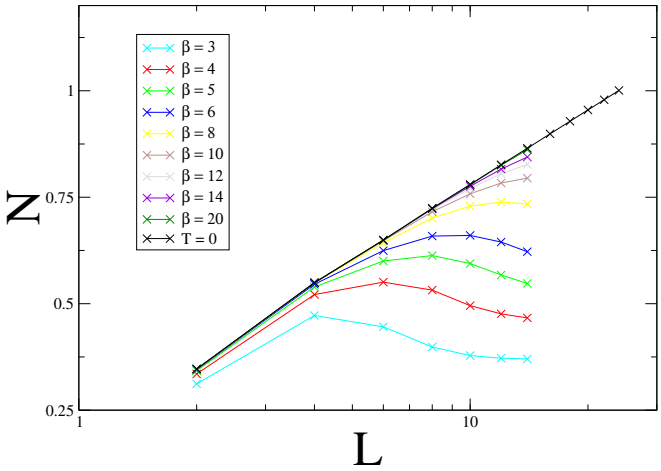

FIG. 6. Logarithmic negativity for the antiferromagnetic Heisenberg model ($\lambda = 1$) as a function of the length L for periodic chains at different β values. Also shown are the results from the Lanczos based calculation for the ground state negativity.

$T \geq 0$, since the entanglement entropies generally did not. The negativity was then computed as a function of the number of sites for various values of β. For values of β sufficiently low, it was observed that the negativity converged to some finite value, with increasing system size. This constant value is indicative of an area law, since the boundary between partition A and B in 1D is simply a point. When $\lambda \geq 1$ the negativity increased logarithmically at first, but then saturated with increasing system size, as long as β was sufficiently small. We believe that the negativity should saturate for any finite β, if the system size is sufficiently large. It was observed that as we increased β the curve never exceeded the logarithmic curve observed in the ground state. A similar calculation was performed for the TFIM, with the system tuned to the quantum critical value of $J_c = 2$. For this value, logarithmic corrections to the area law are needed in the ground state, and again it was observed that at large enough temperature values, true area laws were realized. The observations explained are shown in Fig. 6 and 7.

When studying the true Ising model on a 1D chain, there are degenerate ground states, call them say $|0\rangle$ and $|1\rangle$, such that

$$|0\rangle = |\downarrow\downarrow \ldots \downarrow\rangle \quad ; \quad |1\rangle = |\uparrow\uparrow \ldots \uparrow\rangle$$

for a ferromagnetic model, or the two Neel states for an anti-ferromagnetic model. However, once a small field is applied transverse to the spin direction such as in the TFIM, these two states mix forming two new "Schrodinger Cat" states with a small energy difference. The states are

$$|\pm\rangle \simeq \frac{1}{\sqrt{2}} \left[|0\rangle \pm |1\rangle\right]$$

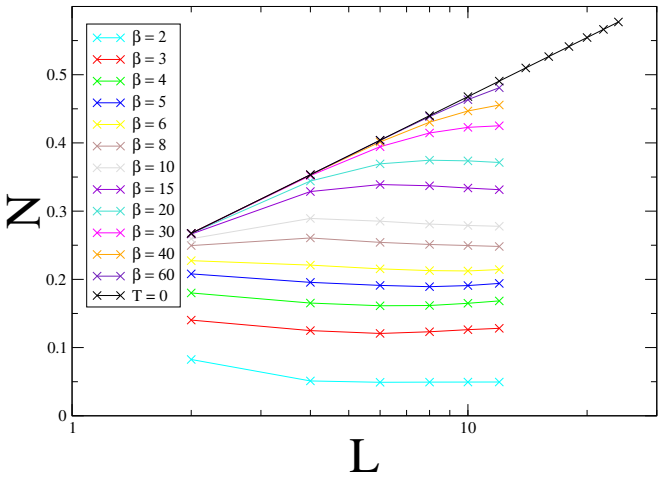

FIG. 7. Logarithmic negativity for the transverse-field Ising model (TFIM) tuned to its quantum critical point ($J = 2$) as a function of the length L for periodic chains at different β values. Also shown are the results from the Lanczos based calculation for the ground state negativity.

with energy difference $\Delta \propto e^{-\frac{L}{\xi}}$ with L the length of the chain. Both states are approximately Bell states, with entanglement approximately $\ln 2$. In practice, these states are not true bell states, and there should be additional entanglement contributions. When considering a thermal state, at moderate temperature scales, these two states are almost degenerate, and they yield a mixed state, with negativity lower than being in just one of the states. To achieve essentially ground state negativity values, the temperature needs to be on the order of Δ yielding very low temperature scales, and the ground state

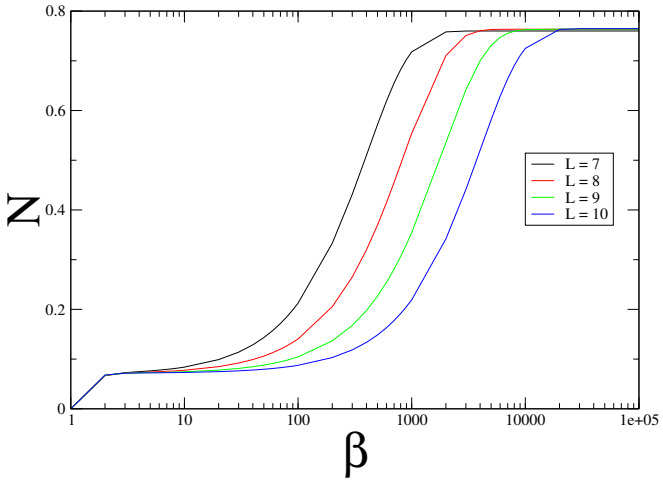

FIG. 8. Logarithmic negativity in chains of different length as a function of temperature in a regime where the ground state is ordered in the thermodynamic limit. In any finite system, the ground state is a Schrodinger Cat state, with a gap to the first excited state which is exponentially small in the system size. This means that the asymptotic $T = 0$ value is obtained only at β values that scale exponentially with size of the system.

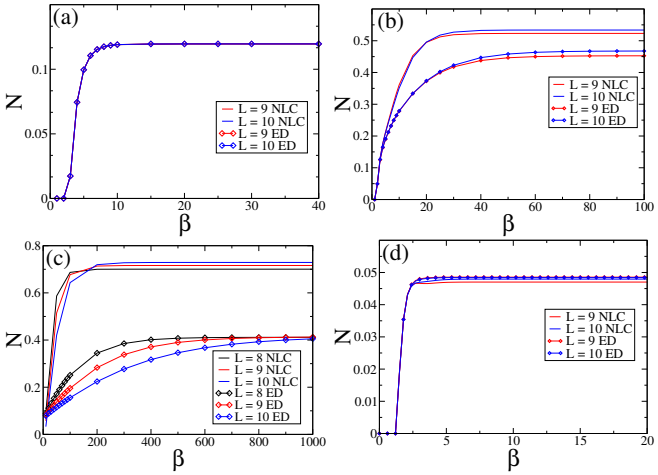

FIG. 9. Comparison between NLC and exact diagonalization (ED) is shown for the transverse-field Ising model (TFIM) for rings of length L. Different values of J were used with (a) $J = 0.5$, (b) $J = 2(J_c)$, (c) $J = 3$, and (d) $J = 3$. In (d), a small parallel field is applied, killing the cat states, showing good agreement between NLC and ED.

negativity is approximately $\ln 2$ as it should be. This is illustrated in Fig. 8 for the TFIM with $J = 4$.

When considering 2D systems, simply applying periodic boundaries was not sufficient for achieving the thermodynamic limit, so a Numerical Linked Cluster (NLC) expansion was employed. Due to the simplicity of the TFIM, we examined this model in 2D. As a check for the validity of NLC, we compared against exact diagonalization (ED) with periodic boundary conditions in 1D. For 1D systems, the results found in the thermodynamic limit when performing ED should be twice that of the results of NLC since there are two boundaries in ED, and entanglement follows an area law.

The comparison between the two methods of NLC and ED are illustrated in Fig. 9. The agreement for small values of J is incredible, with it being nearly impossible to distinguish which curve came from ED and which curve came from NLC, as illustrated in Fig. 9(a). As J gets larger, and approaches $J = 2$ (J_c), the agreement begins to deviate at low temperatures, but in this regime, neither NLC nor ED has truly converged. For $J = 2$ it was seen that they do not match numerically, but being that J_c is 2, and the negativity diverges logarithmically here, there is no guarantee that these values will agree. For $J > 2$ cat states begin to emerge, and the assumption that ED is twice that of NLC is no longer valid, since there is entanglement due to the presence of cat states, and this dominates the entanglement due to interactions at the boundaries. However, if a small magnetic field is applied in the z direction, the cat states can be destroyed, eliminating the entanglement due to cat states, and leaving behind the entanglement due to interactions at the boundary. This is illustrated in Fig. 9(d) with good agreement. It is clear that NLC has not converged yet, but it appears to be approaching the value realized by ED.

Calculations were performed in 2D using NLC for the TFIM. Graph counting for NLC was done so that it yielded the quantity per unit area of the boundary. Therefore, if NLC converged, it implies an area law. There is a known phase transition occurring in the ground state at a value of $J_c \approx 0.657$ in this model, with $J > J_c$ corresponding to an ordered phase in which NLC need not converge. For values of $J < J_c$, in the ground state, we expected convergence, due to other entanglement quantities showing an area law in this region. For finite temperatures, convergence was realized, implying an area law, but as the ground state was approached, NLC was not always converging. We believe that is due to negativity not being a smooth function of temperature, not that negativity does not follow an area law. As temperature changes, eigenvalues of ρ^{Γ_B} can become negative, causing jumps in the negativity. As an illustration, Fig. 10 shows a kink in the negativity for a chain of length four with equal sized partitions. At a value of $\beta \approx 7.7$, there is a kink in the negativity curve, and we also see that at the same value, multiple eigenvalues of ρ^{Γ_B} change sign. Due to the nature of NLC, and summing over a large number of clusters, we believe these kinks are amplified, and this is causing difficulty for the NLC method. These difficulties for NLC yield alternating series, in which Euler transformation (ET) [6] can be

employed to accelerate convergence. After ET, the series appears to have converged, even into the ground state, signifying an area law. Fig. 11 illustrates the negativity as a function of β before and after ET. Also, for $J = 0.1$ and for $J = 0.5$ negativity as a function of order is illustrated in Fig. 12. Again it is seen that negativity is maximized in the ground state, and shows monotonic increase with increasing β. For $J = 0.1$ all values of β are showing clear signs of convergence. For $J = 0.5$ the same monotonic behavior is realized, and convergence is seen for most values of β, with some issues in the ground state that after ET shows convergence.

We are interested in how the entanglement persists to non-zero temperatures, when the phase transition is classical in nature. Quantum entanglement should play no role, and hence we would also expect that negativity should show no signs at the classical critical point. Fig. 13 shows negativity as a function of J for various temperatures up to orders 7 and 8, as well as the expected critical couplings J_c. At $T = 0$, NLC clearly shows a maximum that is getting sharper near the critical point. Because this expansion favors the ordered phase, the peak occurs slightly below J_c, but is getting closer with order. At higher temperatures, negativity is zero below a certain J because of the sudden death phenomenon, and so the peak is shifted to higher J. The expansion appears essentially converged for these curves, and the location of the peak is uncorrelated with the classical transition points, confirming our expectation that negativity plays no role in classical phase transitions. In fact, beyond a certain point, negativity is zero at the classical phase transition. Fig. 14 shows the sudden death temperature for the model, which intersects with the classical phase transition point.

CONCLUSIONS

In conclusion, in this paper we studied the bipartite entanglement negativity for both the TFIM and XXZ Hamiltonians for 1D chains and rings, and the TFIM for a 2D square lattice. Remarkable agreement between ED and NLC in the TFIM for regions with no long range correlations gave us confidence to explore the 2D square lattice using an NLC series expansion. Trouble arose when working in regions in which there were long range correlations, or when the alternations of the series were strong.

The results computed for the ground state matched qualitatively the same structure of the bipartite entanglement entropies, and needed the same logarithmic corrections as those needed for the entropies when in phases with gapless excitations. Field theory results for the Renyi entropies in terms of central charge were satisfied by the negativity with Renyi index $\frac{1}{2}$.

The exploration of thermal states yielded significantly

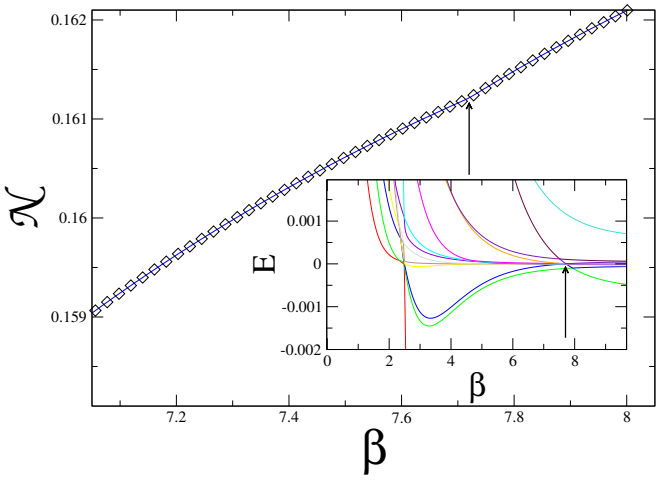

FIG. 10. A second onset in negativity happens for a 4-site system below an inverse temperature of $\beta = 8$. The inset shows eigenvalues of the partially transposed density matrix as a function of β. Note that in a small window near these onsets there are multiple eigenvalues which change sign (going from positive to negative and from negative to positive)

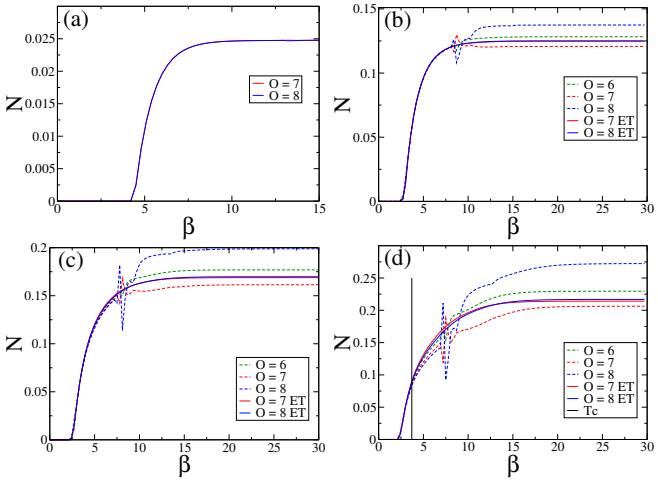

FIG. 11. Logarithmic negativity for the transverse-field Ising model (TFIM) on a square lattice as a function of inverse temperature β, obtained via NLC. Multiple values of J are shown with (a) $J = 0.1$, (b) $J = 0.5$, (c) $J = 0.657(J_c)$, and (d) $J = 0.8$

different results than those of the entropy based entanglement measures. It was observed that in a general manner, the negativity decreased monotonically with increasing temperature, having its maximum value in the ground state. We argue this monotonic behavior reflects the fact that a thermal state arises from strong entanglement with the environment, leaving only limited room for entanglement between different parts of the subsystem. It was shown that the negativity followed an area law in all regions studied, and for systems where logarithmic

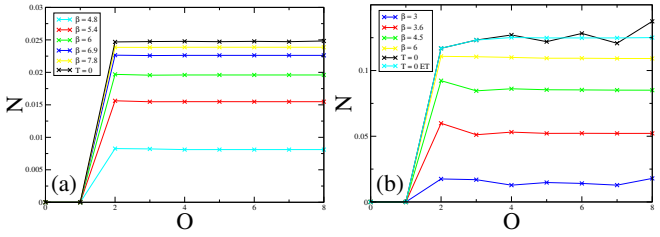

FIG. 12. Negativity as a function of order for the TFIM on a square lattice, with $J = 0.1$ and $J = 0.5$ for (a) and (b) respectively. It is seen that the negativity follows an area law, and again the negativity is maximized in the ground state with monotonic decrease with increasing temperature.

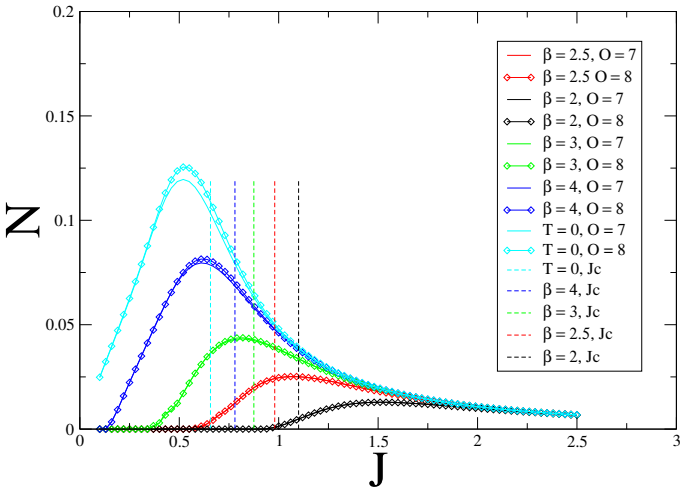

FIG. 13. Logarithmic negativity for the TFIM on a square lattice as a function of J for varying temperatures from the "low temperature" expansions (see text). The O value refers to the order of the NLC calculation.

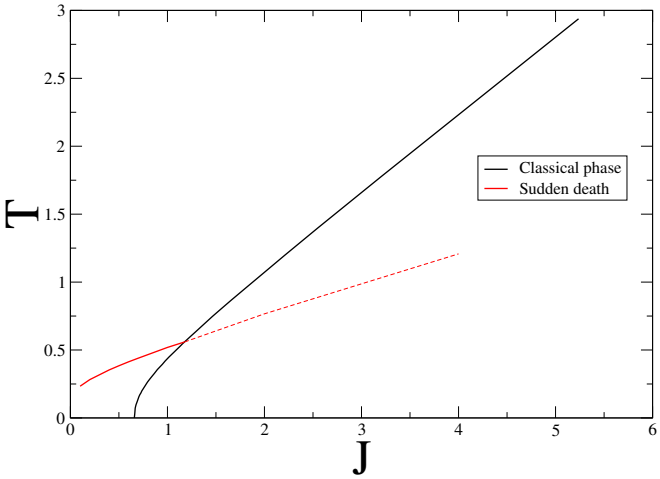

FIG. 14. Values of T above the red curve are in the dead zone, where negativity is zero. It is seen that a large region of the classical phase boundary is in the dead zone. Data for the classical phase boundary gathered from [7]

corrections were needed in the ground state, the negativity showed true area law behavior with sufficiently large temperatures. This was in sharp contrast with the entanglement entropies, as they in general saw volume laws in thermal states.

We found that quantum entanglement does not play a role in classical phase transitions. Logarithmic negativity can be zero at the transition. Or, it can be non-zero with no clear evidence for a singularity in it at the transition. At a T = 0 quantum phase transition, it develops universal critical behavior in temperature and size of the system. The well known log singularity of CFT is rounded off at any nonzero temperature, leaving a logarithmic dependence on inverse temperature β.

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