1. a) Numerator vanishes as \( z \) and denominator vanishes as \( z^2 \), so the singularity has order 1. For the residue, \( \frac{\ln(1-z)}{\sin^2 z} \approx \frac{-z}{(x-\frac{1}{2}z^2)^2} \approx -\frac{1}{z} \), so the residue is -1.

b) Expand the sine to get \( z^2 \sin \frac{1}{z-1} \approx z^2 \left( \frac{1}{1-z} - \frac{1}{6} \left( \frac{1}{1-z} \right)^3 + \ldots \right) \). Clearly the powers of \( \frac{1}{z-1} \) get ever-larger, so this is an essential singularity. For the residue, the sine expansion is already in terms of \( z-1 \), but the outside factor of \( z \) must also be expanded as \( \left( [z - 1] + 1 \right)^2 = (z - 1)^2 + 2(z - 1) + 1 \). The \( \frac{1}{z-1} \) coefficients are \(-1 + \frac{1}{6} = -\frac{5}{6}, \) so this is the residue.

2. a) In \( \int e^{R(-\frac{\pi}{2} + e^{iT} \ln z)} dz \), take \( f(z) = -\frac{z}{R} + e^{iT} \ln z \). The saddle point is where \( f'(z) = -\frac{1}{R} + \frac{e^{iT}}{z} \) vanishes, or \( z_0 = t \). The argument of \( f''(z_0) = -\frac{e^{iT}}{z_0^2} \) is \( \pi - \theta \). (The extra \( \pi \), which many people missed, is from the minus sign in the second derivative.)

Finally, the path over the saddle makes an angle \( \frac{1}{2} \pi - (\pi - \theta) = \theta/2 \) with the positive real axis. For \( \theta = \pi/4 \), the path is at a shallow angle \( \pi/8 \). For \( \theta = 3\pi/4 \), it is at the much steeper angle \( 3\pi/8 \).

b) The integration path starts at the origin, curves upward and to the left, crosses the saddle, and then eventually comes back towards the real axis as \( x \to \infty \). The key point is that there are paths leaving the saddle point perpendicular to the correct integration path, along which the magnitude of the integrand is steadily increasing. The integration path had better avoid these paths (either crossing them or even going close to them), or else the integrand won’t be larger at the saddle point than on the rest of the integration path. The large-magnitude path going off to the left can easily escape to infinity without coming anywhere near the integration path. The large-magnitude path going off to the right can only escape by crossing the positive real axis. This is legit because the positive real axis is where the integral is ultimately desired, but it is NOT part of the (half)-contour where the integrand is assumed to be smaller than at the saddle point.

c) For any \( \theta \), the approximation is \( e^{iT/2}e^{-t+T \ln t} \sqrt{\frac{2\pi}{R^2 t^2}} = e^{iT/2}(e^{-Tt})\sqrt{2\pi R} = \sqrt{2\pi T} \left( \frac{1}{e} \right)^t \), which is exactly the same formula as for real \( t \).

3. Use a long horizontal rectangle that runs along the real axis, then up to the line \( z = x + ib \), then left along that line (backwards from the direction of the integral you want), then back to the real axis. There are no singularities inside this contour, so the integral around the entire loop vanishes. Also in the limit as the horizontal length of the rectangle goes to \( \infty \), the integrand on the vertical sides is \( e^{-a(x+y)^2} = e^{-ax^2+ay^2-2ay} \to 0 \) since \( x \) eventually becomes much larger than any \( y \) on the path. This just leaves the horizontal sides, so \( 0 = \int_{-\infty}^{\infty} e^{-ax^2} dx - \int_{-\infty}^{\infty} e^{-a(x+ib)^2} dx \). The first integral is known, so the second (desired) integral also equals \( \sqrt{\pi/a} \).

4. \( g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{i\omega t} dt \); you can integrate one piece of \( f \) and then get the rest by adding in the Fourier transforms of translates of that piece (with appropriate phase factors for
the translation and scale factors for the different magnitudes). For the piece with \(0 < t < 1\), the integral is

\[
\frac{1}{\sqrt{2\pi}} \int_0^1 \sin \frac{\pi}{2} t e^{i\omega t} \, dt = \frac{1}{\sqrt{2\pi}} \int_0^1 \frac{e^{i\pi t/2} - e^{-i\pi t/2}}{2i} e^{i\omega t} \, dt = \frac{1}{\sqrt{2\pi}} \frac{i\omega e^{i\omega t} + \pi/2}{\omega^2 - (\pi/2)^2}.
\]

The simplification uses the fact that \(e^{i\pi/2} = i\). The different pieces are translated by \(2n\), which introduces a scale factor of \(e^{i2n\omega}\), and also scaled by \((-1)^n/(2n + 1)^2\), where the sign comes because the sine function alternates between being positive and negative. All together this means \(g(\omega) = \sum_{n=-\infty}^{\infty} (-1)^{n+1} \frac{i\omega e^{i\omega t} + \pi/2}{\sqrt{2\pi}(2n+1)^2 \omega^2 - (\pi/2)^2}\).