1. a) To Fourier transform the equation, multiply each term by \( \frac{1}{\sqrt{2\pi}} e^{-ikx} \) and integrate over all \( x \):

\[
-\frac{D}{2} \int_{-\infty}^{\infty} dx \frac{d^2 \phi(x)}{dx^2} e^{-ikx} + \frac{K^2 D}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \phi(x) e^{-ikx} = \frac{Q}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \delta(x) e^{-ikx}.
\]

The integral on the right-hand side becomes \( \frac{Q}{\sqrt{2\pi}} e^{-ikx} \big|_{x=0} = \frac{Q}{\sqrt{2\pi}} \). For the first term on the left-hand side, use the fact that the Fourier transform of a function’s derivative is \(-ik\) times the Fourier transform of the function. For a second derivative, multiply by \(-ik\) twice. \((-ik)^2 = -k^2.\) So, the equation for \(g(k)\) becomes \(Dk^2 g(k) + DK^2 g(k) = \frac{Q}{\sqrt{2\pi}}.\) (Don’t confuse the upper and lower case \(k\)’s!) Solving for \(g(k)\) gives \(g(k) = \frac{Q}{\sqrt{2\pi D (k^2 + K^2)}}.\) You could include a pair of \(\delta\)-functions for the homogeneous solution, but those will disappear in part b).

b) To get this back into real space, use the inverse transform \(\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk g(k) e^{ikx} = \frac{Q}{2\pi D} \int_{-\infty}^{\infty} dk e^{ikx} / k^2 + K^2.\) Now do contour integration, with \(k\) as a complex variable. For \(x > 0\), the desired integral along the real axis equals the limit of \(\frac{Q}{2\pi D} \int_{\text{contour}} dk e^{ikx} / k^2 + K^2\) around a big semicircle in the upper half-plane, as the radius goes to \(\infty.\) There is one pole inside, of order 1, at \(k = iK.\) (I assume \(K > 0.\)) The integral becomes \(2\pi i \frac{Q}{2\pi D} \int_{\text{contour}} dk e^{ikx} / 2ik = \frac{Qe^{-Kx}}{2DK}.\) For \(x < 0\) you need a semicircle contour in the lower half-plane instead, which includes the pole at \(k = -iK\) and gives \(\frac{Qe^{Kx}}{2DK};\) these cases can be combined as \(\frac{Qe^{K|x|}}{2DK}.\) (Remember that you pick up a minus sign when the contour is in the lower half-plane, since the contour traverses the real axis from positive to negative.) Even if you included \(\delta(k \pm iK)\) terms in \(g(k)\) in part a), those terms go away upon doing the integral. (The integrand of the \(\delta\)-functions never vanishes on the integration path.) Another way of looking at this is that the homogeneous solutions are \(e^{\pm kx},\) and neither one of these can enter in if the solution is going to be well behaved as \(|x| \to \infty\) in both directions.

2. The Green’s function satisfies \(G''(t, t') + 2G'(t, t') + G(t, t') = \delta(t - t'),\) where the derivatives are with respect to \(t.\) Away from \(t',\) the solutions are those of the homogeneous differential equation, \(e^{-t}\) and \(te^{-t}.\) (You can find them through the characteristic polynomial.) With the reasonable assumption that nothing happens before the drive, \(G(t, t') = 0\) for \(t < t'.\) We’ll write \(G(t, t') = Ae^{-t} + Bte^{-t}\) for \(t > t'.\) Continuity of the function at \(t'\) gives \(A = -Bt';\) integrating the differential equation in a tiny window near \(t'\) shows that the derivative changes by 1 at \(t'\) and gives a second equation for the coefficients. Solving gives \(B = e^{t'}\) and \(A = -t'e^{t'}.\) Finally, the solution with the given drive can be written as \(f(t) = \int_{-\infty}^{\infty} D(t')G(t, t')dt' = \int_{0}^{\infty} G(t, t')dt'.\) There are three relevant cases to consider. First, for \(t < 0\) the Green’s function vanishes everywhere within the integration region and \(f(t) = 0.\) Second, for \(t > a,\) the integral becomes \(f(t) = -\int_{a}^{\infty} (t' - t)e^{t'-t}dt' = (t + 1 - a)e^{a-t} - (t + 1)e^{-t}.\) Third, for \(0 < t < a\) the integration region will cut off at \(t,\) so \(f(t) = -\int_{0}^{a} (t' - t)e^{t'-t}dt' = 1 - (1 + t)e^{-t}.\) You can check that \(f\) is continuous at both 0 and \(a.\)

Once you have the Green’s function, you can adapt to any other drive term by redoing just the convolution integral.

3. a) Start with the differential equation for the Green’s function. Multiply both sides by \(f(x'),\) integrate over \(x',\) and switch the order of \(A\) and the integration. This results in a solution to the desired equation, \(\psi(x) = \int_{-\infty}^{\infty} G(x, x')f(x')dx'.\)

b) \(G(x, x') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(k, x') e^{ikx} dk\) To find \(G(k, x'),\) transform the equation for \(G(x, x')\) into \(-k^2 G(k, x') - \frac{k^2 D}{\sqrt{2\pi}} G(k, x') - 3G(k, x') = -\frac{Q}{\sqrt{2\pi}} e^{-ikx}.\) (For the signs on the right and in the first-derivative term, note that \(G(k, x') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(x, x') e^{-ikx} dx.\) After solving for \(G(k, x'),\)

we get \(G(x, x') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-ikx'}}{\sqrt{2\pi(k^2 + 2k + 3)}} e^{-ikx} dk.\) (Unlike the example in class, the denominator never vanishes for real \(k,\) so we needn’t worry about the possibility of delta-function spikes.
4. a) These signals look identical because of aliasing. Increasing the measuring time won’t help; the
The first few maxima are at (0.754, 0.017), (1.382, 0.00937), (2.011, 0.00649), (2.639, 0.00498),
and (3.267, 0.00405). Fitting gives $A = 0.013, n = –0.98$.

b) Doubling $a$, for example, will halve the period of the oscillations that show up in the Fourier transform. (That’s because lengths in real space turn into inverse lengths in Fourier space.) Doubling $a$ also doubles the height of the Fourier transform. (That’s necessary because the real-space sum of $|f_k|^2$ doubles, but without the amplitude change the Fourier sum of $|c_p|^2$ would be halved. The extra 2 in the amplitude means the Fourier sum also doubles.) In general increasing $a$ by a factor of $r$ will increase the Fourier amplitude by $r$ and decrease the spacing of the Fourier features by $r$.

c) For $x > x'$, the exponential in the integral has the form $e^{iak}$, with $a > 0$. In this case the integral can be carried out using a large semicircle contour closing in the upper half-plane. The singularities are at $k = -3i$ and $k = i$, with only the latter inside the contour. So
$$G_1(x, x') = (2\pi i)^{-1} e^{iak} e^{-iak} = \frac{1}{4} e^{-(x-x')}.$$ For $x < x'$, the exponential is $e^{iak}$ with $a < 0$, so the integral must be closed in the lower half-plane. Now only the singularity at $k = -3i$ is inside the contour. There is an additional minus sign since this contour follows the real axis from right to left. All told, $G_2(x, x') = \frac{1}{4} e^{3(x-x')}.$

d) Away from $x = x'$, $G(x, x')$ must be a homogeneous solution of the differential equation. The general homogeneous solution is $Ce^{-x} + De^{3x}$. (Plug in $e^{i\alpha x}$ and solve for $\alpha$.) For sanity at $\pm \infty$ (i.e., the Green’s function should remain finite), we need $C = 0$ as $x \to -\infty$ and $D = 0$ as $x \to \infty$. This is indeed achieved in part c), since $x \to -\infty$ corresponds to $G_2$ and $x \to \infty$ corresponds to $G_1$.

Approaching $x'$ from above, $G(x, x')$ goes as $e^{-x}$, with derivative going as $-e^{-x}$. Approaching from below, $G(x, x')$ goes as $e^{3x}$, with derivative going as $3e^{3x}$. The function has a peak at $x'$.

5. a) The complete overlay of real-space graphs happens when the total time is the same for both. Here that requires $dt = 150/NMAX$.

b) Total length in Fourier space corresponds to the time interval in real space, so constant Fourier length just means a constant $dt$, independent of $NMAX$. Constant frequency spacing corresponds to length (i.e., total time) in real space, which from part a) means $dt = 150/NMAX$.

c) For $x > x'$, the exponential in the integral has the form $e^{iak}$, with $a > 0$. In this case the integral can be carried out using a large semicircle contour closing in the upper half-plane. The singularities are at $k = -3i$ and $k = i$, with only the latter inside the contour. So
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b) This time the trouble is not that the original measured points are identical, but that they’re too close. Over 10 seconds, the length of the original measurement, $\sin 5\pi t$ completes 25 cycles and $\sin 5.05\pi t$ completes $25\frac{1}{4}$ cycles. Given enough time, these waves will eventually look very different in spots, but 10 seconds isn’t quite long enough. So double the measurement time, taking points at $t = 0.01n$ seconds ($0 \leq n \leq 1999$). At 20 seconds, the signals will be exactly out of phase (since the frequency difference is 0.05=1/20), so the difference will be large at the last maximum-amplitude spot. $\sin 5\pi t$ has a minimum at $19.9=(0.01)(1990)$ seconds. For $n = 1990$, $\sin 5\pi (19.9) = -\sin 195\pi (0.015) = -1.9999 \mu V.$
b) Repeat for a triangle pulse of height 1 and half-width 10. Maxima: (0.880, 0.00312), (1.550, 0.00110), (2.178, 0.000558), (2.806, 0.000337), and (3.435, 0.000226). $A = 0.0025$, $n = -1.88$.

c) Maxima: (2.890, 1.375 × 10−5), (4.189, 4.713 × 10−6), (5.445, 2.145 × 10−6), (6.660, 1.123 × 10−6), (7.959, 6.860 × 10−6). $A = 0.00031$, $n = -2.92$.

The $n$ values would be integers if things were perfect. They aren’t because, for example, the maximum point picked out in the discrete Fourier transform needn’t be the exact maximum of its lobe. The key point is how many continuous derivatives a function has, and the spot with the least continuity determines the falloff in Fourier space. If you believe that a discontinuous function (like the square wave) has a Fourier transform that falls off as $1/\omega$, the rest follows. The derivative of the triangle wave is discontinuous, so its Fourier transform falls off as $1/\omega$. But the Fourier transform of the derivative of the triangle wave goes as $\omega$ times the Fourier transform of the triangle wave itself, so the latter must fall off as $1/\omega^2$. You get another factor of $1/\omega$ for the parabolas, where the first derivative is continuous but the second derivative is not.

7. a) The discrete Fourier transform assumes that functions are periodic, in this case with period $NMAX * dt = 150$. That requires frequencies that are integer multiples of $2\pi/150 \approx 0.0418879$. The frequency given for this problem is not an appropriate multiple, which means that the real-space function is not actually a sine wave. It’s actually the absolute value of a sine function. And since it isn’t a sine wave, the peaks in its Fourier transform aren’t delta functions.

b) A couple of things happen here. First, the real-space function is no longer symmetric, so the Fourier transform is not real. In fact at low frequencies the imaginary part of the Fourier transform is much larger than the real part. So for a better comparison look at the amplitude. The original peak is much sharper; for example, the amplitude at frequency 0.20944 is 15% of the amplitude at 0.0418879 for the new function, but only 3% for the original function. This effect is the difference between the Fourier transform of a continuous function with discontinuous derivative, which falls off as $\omega^{-2}$, and the Fourier transform of a discontinuous function, which falls off more slowly as $\omega^{-1}$.

c) As noted in part a), to get a single peak we need $w$ to be an integer multiple of 0.0418879. Since 10/0.0418879 ≈ 239, take $w = 10.0112$.

Optional 1. The charge distribution is the inverse transform of $\rho$, $\rho(r) = \frac{1}{(2\pi)^3} \int 0^\infty k^2 dk j_0^\pi \sin \theta d\theta j_0^2 \pi d\phi \frac{a^2 e^{ikr \cos \theta}}{a^2 + k^2} = \frac{1}{(2\pi)^3} \int 0^\infty k^2 dk \int 0^\infty e^{ik \cos \theta} a^2 e^{-ikr} dy$. Choose $k$-space coordinates so that $k_z$ is aligned with $r$. This means your $k$-space coordinates will be different for different values of $r$, but that’s fine—you’re only setting the coordinates so you can do the integral. Then $\rho(r) = \frac{1}{(2\pi)^3} \int 0^\infty k^2 dk j_0^\pi \sin \theta d\theta j_0^2 \pi d\phi \frac{a^2 e^{ikr \cos \theta}}{a^2 + k^2} = \frac{1}{(2\pi)^3} \int 0^\infty k^2 dk \int 0^\infty e^{ikr \cos \theta} a^2 e^{-ikr} dy$. Use a semicircle closing in the upper half-plane, since $r$ is always positive. There’s one enclosed singularity at $k = ia$, and the final result is $\rho(r) = \frac{a^2}{kr} e^{-ar}$. I originally wrote the argument of $\rho$ as a vector, but the final result shows that the charge distribution is indeed spherically symmetric.

Optional 2. a) There are all sorts of possibilities here. On these four points, sin $x$ becomes (0,1,0,-1). The function

$$f(x) = \begin{cases} 1 & \pi/16 \leq x \leq 25\pi/16 \\ 0 & \text{otherwise} \end{cases}$$

is clearly not orthogonal to sin $x$ (the relevant integral is definitely positive), but the corresponding vector (0,1,0,1) is orthogonal to (0,1,0,-1).
b) Again, all sorts of options. For example use a function which is non-zero on an interval including \( \pi/2 \), and also non-zero on an interval between \( \pi \) and \( 2\pi/2 \), with the widths and magnitudes chosen so \( \int f(x) \sin x \) vanishes. One such function is

\[
f(x) = \begin{cases} 
1 & \pi/4 < x < 3\pi/4 \\
\frac{\sqrt{2}}{\sqrt{3}-1} & 7\pi/6 < x < 4\pi/3 \\
0 & \text{otherwise}
\end{cases}
\]

Here I picked the endpoints so that after doing the above integral I could evaluate cosine without a calculator, and I picked the weird amplitude in the third quadrant to make the function orthogonal to \( \sin x \). In the discrete case, \( f \) becomes \((0,1,0,0)\) and the dot product is 1.

**Optional 3.** There are two considerations. First, to reach a frequency of 402 Hz (or a tad higher, if the frequencies can vary slightly), you need to sample a minimum of 2 points per 402-Hz cycle, or \( dt < 1/2(402) = 0.00124 \) seconds. Second, in order to resolve frequency differences of 0.02 Hz, you need to measure for at least \( 1/.02 = 50 \) seconds. Combined with the maximum on \( dt \), this requires at least 40,323 data points. (If you’re getting different numbers, it might be units: Hz measures \( f \) rather than \( \omega \), and \( \omega = 2\pi f \).)

**Optional 4.** The transform in 6b (triangle pulse) should be closest to the parabola. Both are continuous but with discontinuous first derivatives, so their long-range (high-frequency) behavior has the same overall frequency dependence.