Riley 24.21 Consider \( \int (\ln z)^2 \frac{1}{1+z^2} \, dz \) over the path specified. The only singularity inside the contour is at \( z = i \), so the integral becomes \( 2\pi i \frac{(\ln i)^2}{2i} = -\pi^3/4 \). On the large semicircle, parametrize with \( z = Re^{i\theta} \). The integrand magnitude approaches \( \frac{(\ln R)^2}{R} \), where one factor in the denominator is cancelled by the \( R \) in \( dz \) (which corresponds to the increasing length of the path as \( R \) grows). This magnitude approaches zero for large \( R \), so the contribution of the large semicircle vanishes. Similarly, on the small semicircle parametrize with \( z = \varepsilon e^{i\theta} \), and the integrand magnitude approaches \( (\ln \varepsilon)^2 \varepsilon \), which goes to zero as \( \varepsilon \to 0 \). (Even though the original function blows up at zero, the path is disappearing even faster.) This leaves the two segments through a large-amplitude region. Then between the origin...but it’s an ugly essential sin-

Riley 25.22 The \( C_1 \) integral starts on the positive \( x \)-axis, somewhere near the origin, goes through \( z = i \), and then heads towards the negative \( x \)-axis and goes off to \( -\infty \). Take \( f(t) = t - \frac{1}{t} \). Then \( f'(t) = 1 + \frac{1}{t^2} \) is zero at \( \pm i \). Only \( i \) is on curve \( C_1 \). Here \( f''(i) = 2/i \), which has argument \(-\pi/2 \). That gives \( \alpha = \frac{\pi}{2} - \frac{1}{2}(-\frac{\pi}{2}) = \frac{3}{4}\pi \), which is in the correct general direction (right-to-left). The saddle contribution is \( \frac{1}{2\pi i} e^{\frac{i\pi}{4}}(i-\frac{1}{4})e^{\frac{i\pi}{2}} \sqrt{\frac{2\pi}{3(\theta)}} \). When simplifying, note that \( \frac{i^{-1}}{4} = -1 \), and incorporate this factor of -1 by changing \( e^{i\frac{\pi}{2}} \) to \( e^{-i\pi/4} \). The remaining \( i^{-\nu} \) can be written \( e^{-i\pi
u}/2 \). Also multiply by 2 overall, to get
\[
H_\nu^1 \to \sqrt{\frac{2\pi}{3\nu}} e^{\nu(z-\nu)/2-\pi/4}.
\]
For \( C_2 \), we must end up with \( \alpha = \pi/4 \), and \( i \) changes to \(-i \) in the factors \((-i)^{-\nu+1} e^{-iz} \). This gives \( H_\nu^2 \to \sqrt{\frac{2\pi}{3\nu}} e^{i(z-\nu)/2-\nu/4} \).

Since \( J_\nu(z) = \frac{1}{2} \left( H_\nu^1(z) + H_\nu^2(z) \right) \), \( J_\nu(z) \to \sqrt{\frac{2\pi}{3\nu}} \cos(z - \frac{\nu\pi}{2} - \frac{\nu}{4}) \).

Note that the combined integral about \( C_1 \) and \( C_2 \) doesn’t vanish because of the singularity at the origin. In principle for the asymptotic form of \( J \) one could skip the steepest descent business and instead evaluate the residue of \( g \) at the origin...but it’s an ugly essential singularity, so that’s a lot easier said than done. It’s important that the contour has a cusp at the singularity, approaching and leaving it horizontally. That means that as far as the residue theorem is concerned, the singularity acts as inside (not on) the contour.

Riley 25.23 Take \( f(t) = -iz \sin t + i\nu t \), so \( f'(t) = -iz \cos t + i\nu \) vanishes at \( t_0 = \arccos(\nu/z) \).

This means an infinite set of saddle points all on the real axis, but we’ll aim for the one between \(-\pi \) and 0. Any of the others would involve a distorted path that has to pass through a large-amplitude region. Then \( f''(t_0) = iz \sin(\arccos(\nu/z)) = -iz \sqrt{1 - \frac{\nu^2}{z^2}} \), which has argument \( 3\pi/2 \). (I use the negative square root because I’m requiring \(-\pi < t_0 < 0 \), and sine is negative in this region.) This gives \( \alpha = \pi/2 - (3\pi/4) = -\pi/4 \), which is good because it involves going from the upper half-plane to the lower half-plane when crossing the saddle. (If you use \(-\pi/2 \) as the argument of \( f'' \), you get \( \alpha = 3\pi/4 \) and then have to adjust the
direction by $\pi$.) Now plug into the saddle contribution formula: $e^{iz\sin t_o + it_o}e^{-i\pi/4}\sqrt{\frac{2\pi}{z^2 - \nu^2}}$. For very large $z$, $\nu/z \to 0$ and $t_o \to \pi/2$. I'll also replace $\sqrt{z^2 - \nu^2}$ by $z$. The above expression becomes $\sqrt{\frac{2\pi}{z^2}}e^{i(z+\frac{\pi}{4} - \frac{\nu}{z})}$.

1. The pole at $z = -1$ is within the contour. It has order 2, so calculate the residue by $(z^a)|_{z=-1} = a(-1)^{a-1} = ae^{i\pi(a-1)} = -ae^{i\pi a}$. The integral on the big circle vanishes as $R \to \infty$, and the integral on the small circle around the origin vanishes as $\delta \to 0$. This leaves $-2\pi i e^{i\pi a} \int_0^\infty \frac{e^{i\pi a}e^{x}}{(x+1)^2}dx = -\frac{2\pi i e^{i\pi a}}{1-e^{2\pi a}} = \frac{\pi a}{\sin \pi a}$.

2. a) As $|z| \to \infty$ the integrand looks like $z^{-5/3}$. The circumference of the big circle is proportional to $|z|$, so the integral on the large circle vanishes. As $|z| \to 0$ the integrand looks like $z^{1/3}$, and the small circle’s circumference also vanishes, so that integral vanishes.

b) $\int_0^\infty \cos x^2 dx = \Re\int_0^\infty e^{ix^2} dx$. Integrate $\int e^{ix^2} dx$ on a wedge of angle $\frac{\pi}{4}$. There are no singularities within the contour, so the contour integral vanishes. Parametrizing each side, $0 = \int_0^R e^{ix^2} + e^{iR^2e^{i\theta}} + \int_0^\theta R e^{i\theta} d\theta = \int_0^R e^{ix^2} e^{i\alpha/2} e^{i\pi x^2/4} dx$. In the limit as $R \to \infty$, the integral on the large arc vanishes because the real part of the exponential goes to 0 faster than the extra factor of $R$ blows up. Finally, the integral along the slanted side of the wedge becomes $\int_0^\infty e^{-x^2/2} dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$. Hence $\int_0^\infty \cos x^2 dx = \sqrt{\frac{\pi}{2\sqrt{2}}}$. (From the imaginary part, it turns out that $\int_0^\infty \sin x^2 dx$ has exactly the same value.)

3. a) Let $z = Re^{i\theta}$. Then the integrand becomes $e^{i\theta(\frac{1}{2}R^2e^{i5\theta} + Re^{i\theta})} = e^{i\theta(\frac{1}{2}R^2\sin 5\theta + R \cos \theta)}e^{-i\theta(\frac{1}{2}R^2 \cos 5\theta + R \sin \theta)}$. As $R$ gets huge the sign of $\sin 5\theta$ determines whether the exponential is huge or nearly zero. The trouble is that $\sin 5\theta$ has both positive and negative portions in both half-planes, and thanks to the positive portions neither integral will converge.

b) Use function $f(z) = i(\frac{1}{2}z^5 + z)$. Then $f'(z) = i(z^4 + 1)$, and the second derivative is $f''(z) = 4iz^3$. The zeroes of $f'$ and other relevant calculations for each one are shown in the table. Note that some of these quantities could be written differently; for example, in the first line I could have used $-3\pi/4$ instead of $5\pi/4$ for the argument of the derivative. This might change $\alpha$ by $\pi$, but for the time being that doesn't matter because we’re just looking at the angle the path takes in going over the saddle.

<table>
<thead>
<tr>
<th>zero of $f'$, $z_o$</th>
<th>$f''(z_o)$</th>
<th>$\arg(f'(z_o))$</th>
<th>$\alpha$</th>
<th>$f(z_o)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^{i\pi/4}$</td>
<td>$4ie^{i\pi/4}$</td>
<td>$5\pi/4$</td>
<td>$-\pi/8$</td>
<td>$i(\frac{1}{2}e^{i5\pi/4} + e^{i\pi/4})$</td>
</tr>
<tr>
<td>$e^{3i\pi/4}$</td>
<td>$4ie^{3i\pi/4}$</td>
<td>$3\pi/4$</td>
<td>$\pi/8$</td>
<td>$i(\frac{1}{2}e^{-i\pi/4} + e^{i3\pi/4})$</td>
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<td>$i(\frac{1}{2}e^{-i5\pi/4} + e^{-i\pi/4})$</td>
</tr>
</tbody>
</table>

The good approximation is with a path through the two saddles in the upper half-plane. It goes over the left saddle at the fairly shallow angle of $\pi/8$, then over the right saddle at $-\pi/8$. Avoiding either of these saddles requires passing through a higher-amplitude region to one side or the other. Similarly, getting from the negative real axis to the saddle at $e^{i5\pi/4}$ requires crossing a path of increasing amplitude that begins at the saddle and initially goes at an angle of $7\pi/8$. The integrand amplitude along the real axis is 1, so for large enough $s$ the increasing-amplitude path from the
lower saddle always stays below the axis. The same problem arises in getting from the saddle at $e^{i\pi/4}$ to the positive real axis.

c) Calculate contributions from the two saddle points separately and add them together.

The left saddle gives

$$\frac{\sqrt{2\pi} e^{i\pi/8} (\frac{1}{2} e^{-i\pi/4} + e^{i\pi/4}) e^{i\pi/8}}{2\sqrt{s}} = \sqrt{\frac{\pi}{2s}} e^{is\left(\frac{1}{2} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) e^{i\pi/8}}$$

$$= \sqrt{\frac{\pi}{2s}} e^{is\left(\frac{1}{2} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) e^{i\pi/8}}$$

and similarly the right saddle gives

$$\sqrt{\frac{\pi}{2s}} e^{-is\left(\frac{1}{2} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) e^{-i\pi/8}} = \sqrt{\frac{\pi}{2s}} e^{-is\left(\frac{1}{2} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) e^{-i\pi/8}}.$$ 

Adding these gives

$$\sqrt{\frac{\pi}{2s}} e^{-as\left(2\cos\left(as - \frac{\pi}{8}\right)\right)} = \sqrt{\frac{\pi}{2s}} e^{-as\cos\left(as - \frac{\pi}{8}\right)}.$$ 

Here $a = \frac{4}{5\sqrt{2}}.$